A Finite Memory Interacting Pólya Contagion Network and its Approximating Dynamical Systems

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1 Introduction

2 Finite Memory Interacting Pólya urn Network

3 Stochastic Properties

4 Approximating Dynamical Systems

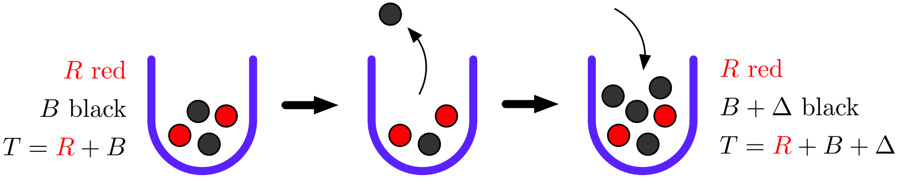
5 Simulation Results

A new model for spread of infection

Objective of the talk:

to use **a network of finite memory Pólya urns** to model the spread of infection in a population

Classical Pólya urn Model



At each time t, we add Δ **red** balls or Δ **black** balls if a red ball or a black ball is drawn respectively.

$$Z_t = \begin{cases} 1 & \text{if a } \textbf{red} \text{ ball is drawn at time } t \\ 0 & \text{if a } \textbf{black} \text{ ball is drawn at time } t. \end{cases}$$

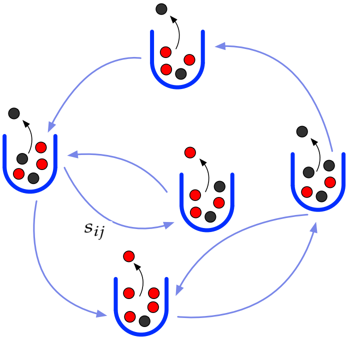
Z_t is the drawing random variable.

Interacting Pólya urn Network

Consider a **network of N Pólya urns**:

$$Z_{i,t} = \begin{cases} 1 & \text{w.p.} \quad \sum_{j=1}^{N} s_{ij} U_{j,t-1} \\ 0 & \text{w.p.} \quad 1 - \sum_{j=1}^{N} s_{ij} U_{j,t-1}, \end{cases}$$

where $U_{j,t-1}$ is the *ratio of* **red** *balls in urn j at time $t-1$.*

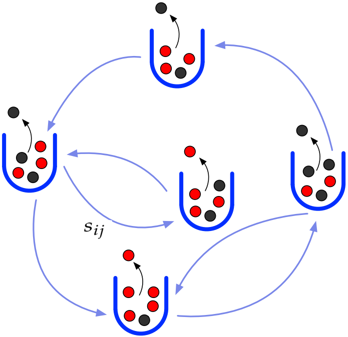


Interconnections prescribed by a **directed graph** with **a row-stochastic adjacency matrix** ($s_{ij} \geq 0$ and $\sum_{j=1}^{N} s_{ij} = 1$ for all $i, j \in \{1, 2, \cdots, N\}$)

the interactions accounts for spatial infection among neighbours

Interacting Pólya urn Network

- At every time instant t, a ball is drawn for every urn.
- if a **red** ball is drawn for urn i at time t, we add $\Delta_{r,i}(t)$ red balls to the urn i at time t.
- if a **black** ball is drawn for urn i, we add $\Delta_{b,i}(t)$ black balls to urn i at time t.



The red balls account for the "infection" in urns and black balls account for the "healthiness" in urns.

Interacting Pólya urn Network

The draw variables $\{Z_{i,t}\}$'s are **conditionally independent given all past draws** in the Interacting Pólya urn Network i.e.,

$$P(Z_{1,t},\dots,Z_{N,t}|\{Z_{1,k}\}_{k=1}^{t-1},\dots,\{Z_{N,k}\}_{k=1}^{t-1}) = \prod_{i=1}^{N} P\left(Z_{i,t}|\{Z_{1,k}\}_{k=1}^{t-1},\dots,\{Z_{N,k}\}_{k=1}^{t-1}\right).$$

We also can compute the **ratio of red balls** at each urn at a given time:

$$U_{i,t} = \frac{X_{i,t-1}}{X_{i,t}} U_{i,t-1} + \frac{Z_{i,t}\Delta_{r,i}(t)}{X_{i,t}},$$

where $X_{i,t} := T_i + \sum_{n=1}^{t} Z_{i,n}\Delta_{r,i}(n) + (1 - Z_{i,n})\Delta_{b,i}(n)$ and T_i is the total number of balls in urn i at time $t = 0$.

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Some key observations

- An important characteristic of most reinforcement processes generated via urn models is that they are **non-Markovian**
- The composition of each urn at any given time **affects its composition at every time instant thereafter**
- This property is not realistic when modelling the spread of infection as **one should account for the possibility that infection is cured**

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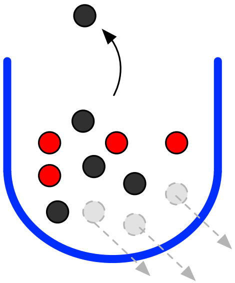
Finite Memory

Interacting Pólya urn Network with N urns and **finite memory** $M \geq 1$ (denoted by $\mathrm{IPCN}(M, N)$):

> *At time instant $t > M$, the reinforcing balls added at time $t - M$ are removed from the urn and hence have no effect on future draws.*



Some parameters for $\mathrm{IPCN}(M, N)$

Ratio of red balls in urn i at time t can be written as

$$U_{i,t} = \frac{\rho_i + \sum_{n=t-M+1}^{t} \delta_{r,i}(n) Z_{i,n}}{1 + \sum_{n=t-M+1}^{t} \left(\delta_{r,i}(n) Z_{i,n} + \delta_{b,i}(n)(1 - Z_{i,n})\right)}, \tag{1}$$

where the parameters defined below will be used later on:

$$\rho_i = \frac{R_i}{T_i}, \qquad \delta_{r,i}(t) = \frac{\Delta_{r,i}(t)}{T_i}, \quad \delta_{b,i}(t) = \frac{\Delta_{b,i}(t)}{T_i}.$$

$U_{i,t}$ is the ratio of red balls in urn i at time t.

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Stochastic Properties

We define $Z_t = (Z_{1,t}, \cdots, Z_{N,t})$

Note that $\{Z_t\}_{t=1}^{\infty}$ is **a time-varying Mth order Markov chain**, and

$$P[Z_{t+1} = a_{t+1} | Z_t = a_t, \cdots, Z_{t-M+1} = a_{t-M+1}]$$
$$= \prod_{i=1}^{N} \left((2a_{i,t+1} - 1) \sum_{j=1}^{N} \frac{s_{ij}\left(\rho_i + \sum_{n=t-M+1}^{t} \delta_{r,i}(n) a_{i,n}\right)}{1 + \sum_{n=t-M+1}^{t} (\delta_{r,i}(n) a_{i,n} + \delta_{b,i}(n)(1 - a_{i,n}))} + (1 - a_{i,t+1}) \right) \quad (2)$$

Even though we can compute the evaluations of this process using this master formula, analysis of this process is complex

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Homogeneous Case

for all i and t:

$R_i = R$, $T_i = T$ and depending on the value of draw variable at time t, $\Delta_{r,i}(t) = \Delta > 0$ or $\Delta_{b,i}(t) = \Delta > 0$

Here $\{Z_t\}_{t=1}^{\infty}$ is a **time-invariant** M**th order Markov chain** and we can draw some conclusions about this process

We define $W_t := (Z_t, Z_{t+1} \cdots, Z_{t+M-1})$. Since $\{Z_t\}_{t=1}^{\infty}$ is an Mth order Markov chain, $\{W_t\}_{t=1}^{\infty}$ forms a Markov chain of order one.

Lemma. For the homogeneous $\mathrm{IPCN}(M, N)$, the Markov process $\{W_t\}_{t=1}^{\infty}$ is irreducible and aperiodic.

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Lemma. For the homogeneous $\mathrm{IPCN}(M, N)$, the Markov process $\{W_t\}_{t=1}^{\infty}$ is irreducible and aperiodic.

Transition probability matrix for homogeneous case

For $M = 1$, the probability of going from state $a = (a_{1,t}, \cdots, a_{N,t})$ to state $b = (b_{1,t+1}, \cdots, b_{N,t+1})$ can expressed as

$$q_{ab}^{(1,N)} := q_{ab}^{(1)} q_{ab}^{(2)} \cdots q_{ab}^{(N)}$$

where

$$q_{ab}^{(d)} = \begin{cases} \frac{\sigma + (1 - \sum\limits_{k=1}^{N} s_{dk} a_{k,t})\delta}{1+\delta} & \text{if} \quad b_{d,t+1} = 0 \\ \frac{\rho + \sum\limits_{k=1}^{N} s_{dk} a_{k,t} \delta}{1+\delta} & \text{if} \quad b_{d,t+1} = 1, \end{cases} \tag{3}$$

where $d \in \{1, \cdots, N\}$ and $\sigma = 1 - \rho$.

Transition probability matrix for homogeneous case

For $M > 1$, if the transition probability of going from state $a = ((a_{11}, a_{21}, \cdots, a_{N1}), \cdots, (a_{1M}, a_{2M}, \cdots, a_{NM}))$ to state $b = ((b_{11}, b_{21}, \cdots, b_{N1}), \cdots, (b_{1M}, b_{2M}, \cdots, b_{NM}))$ is nonzero, it is given by

$$q_{ab}^{(M,N)} := \tilde{q}_{ab}^{(1)} \tilde{q}_{ab}^{(2)} \cdots \tilde{q}_{ab}^{(N)}$$

where

$$\tilde{q}_{ab}^{(d)} = \begin{cases} \frac{\sigma + \left(M - \sum_{i=1}^{N} s_{di} (\sum_{k=1}^{M} a_{ik}) \right) \delta}{1 + M\delta} & \text{if} \quad b_{dM} = 0 \\ \\ \frac{\rho + \left(\sum_{i=1}^{N} s_{di} (\sum_{k=1}^{M} a_{ik}) \right) \delta}{1 + M\delta} & \text{if} \quad b_{dM} = 1, \end{cases} \tag{4}$$

where $d \in \{1, \cdots, N\}$ and $\sigma = 1 - \rho$.

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Approximation by dynamical systems

- The Markov chain $\{W_t\}_{t=1}^{\infty}$ for $IPCN(M, N)$ (not necessarily homogeneous from now on) has 2^{MN} states.
- Due to this exponential increase in the size of the transition probability matrix with the number of urns N and memory M, it is difficult to analytically solve for the stationary distribution in terms of the system parameters

This issue should is also present in the classical SIS/SIR model

We introduce **a class of dynamical systems whose trajectory approximates the** $P(Z_{i,t} = 1)$ at time t for any urn i in the network.

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Mean-field Approximations

At any given time t, we use the following **mean-field approximations**

- For $M = 1$,

$$P(Z_{1,t-1} = a_{11}, \cdots, Z_{N,t-1} = a_{N1}) \approx \prod_{j=1}^{N} P(Z_{j,t-1} = a_{j1}). \quad (5)$$

- For $M > 1$,

$$P[\widetilde{Z}_1 = \underline{a}_{1M}, \cdots, \widetilde{Z}_N = \underline{a}_{NM}] \quad (6)$$
$$\approx \prod_{j=1}^{N} \prod_{k=1}^{M} [a_{jk} P_j(t-k) + (1 - a_{jk})(1 - P_j(t-k))],$$

where $\underline{a}_{iM} = (a_{i1}, a_{i2}, \cdots, a_{iM})$, $\widetilde{Z}_{i,t-1} = (Z_{i,t-1}, Z_{i,t-2}, \cdots, Z_{i,t-M})$

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Approximation by dynamical systems: $M = 1$

We first define:

$$P_i(t) := P(Z_{i,t} = 1),$$

We have that

$$P_i(t) = \sum_{B_{N,1}} P(Z_{i,t} = 1 | Z_{1,t-1} = a_{11}, \cdots, Z_{N,t-1} = a_{N1}) \times P(Z_{1,t-1} = a_{11}, \cdots, Z_{N,t-1} = a_{N1}) \quad (7)$$

where

$$B_{N,1} := \{(a_{11}, a_{21}, \cdots, a_{N1}) \mid a_{i,1} \in \{0,1\} \quad \text{for} \quad i \in \{1, 2, \cdots, N\}\}. \quad (8)$$

Using *mean-field approximation*, we obtain

$$P_i(t) \approx \sum_{B_{N,1}} (P(Z_{i,t} = 1 | Z_{1,t-1} = a_{11}, \cdots, Z_{N,t-1} = a_{N1}) \prod_{j=1}^{N} P(Z_{j,t-1} = a_{j1})) \quad (9)$$

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Approximation by dynamical systems: $M = 1$

Theorem. For the $\mathrm{IPCN}(1, N)$ system, the infection vector given by $P(t) := [P_1(t), P_2(t), \cdots, P_N(t)]^{\mathsf{T}}$ satisfies the equation

$$P(t) \approx J_{N,1} P(t-1) + C_{N,1} \tag{10}$$

where $J_{N,1} \in \mathbb{R}^{N \times N}$, $C_{N,1} \in \mathbb{R}^{N \times 1}$ are matrices with respective entries:

$$[J_{N,1}]_{i \times j} = \frac{s_{ij}(\rho_j + \delta_{r,j})}{(1+\delta_{r,j})} - \frac{s_{ij}\rho_j}{(1+\delta_{b,j})} = s_{ij}(\beta_1^{(j)}(1) - \beta_1^{(j)}(0))$$

$$\text{and} \quad [C_{N,1}]_{1 \times i} = \sum_{j=1}^{N} \frac{s_{ij}\rho_j}{(1+\delta_{b,j})} = \sum_{j=1}^{N} s_{ij}\beta_1^{(j)}(0).$$

The approximating dynamical system for $M = 1$ is linear

Convergence properties: $M = 1$

Theorem. The approximating linear dynamical system for the $\mathrm{IPCN}(1, N)$ system given by (10) has a unique equilibrium point given by $P^* = (I - J_{N,1})^{-1} C_{N,1}$ and

$$\lim_{t \to \infty} P_i(t) = P_i^*$$

for all $i \in \{1, \ldots, N\}$.

In the homogeneous case, we obtain an expected result:

Corollary. For a homogeneous $\mathrm{IPCN}(1, N)$ system, the equilibrium of (10) is given by $P^* = \rho 1_N$, where 1_N is vector of ones of size N and ρ is the ratio of red balls in the urns.

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Approximation by dynamical systems: $M > 1$

Theorem. For the $\mathrm{IPCN}(M, N)$ system, the approximating dynamical system can be written as

$$P_i(t) \approx \sum_{j=1}^{N} s_{ij}\beta_M^{(j)}(0) + \sum_{j=1}^{N}\sum_{n=1}^{M}\Bigg[\Bigg(\sum_{k=0}^{n}\Bigg((-1)^{n-k}\binom{n}{k}s_{ij}\beta_M^{(j)}(k)\Bigg)\Bigg) \times \Bigg(\sum_{\substack{(d_1,\cdots,d_n)\\ \in H_{n,M}}} P_j(t-d_1)\cdots P_j(t-d_n)\Bigg)\Bigg],$$

where

$$H_{n,M} := \{(d_1, d_2, \cdots, d_n) \big| d_i \in \{1, \cdots, M\},\ d_i \neq d_j\ \forall i, j \in \{1, \cdots, n\}\}.$$

The approximating dynamical system for $M > 1$ is highly nonlinear

Linearization for the approximation dynamical systems: $M > 1$

The dynamical system that we have obtained for $M > 1$ are complex to analyze

we further linearize the dynamical systems for the case $M > 1$

Corollary. The linear part of the dynamical system is given by

$$P_i(t) \approx \sum_{j=1}^{N} s_{ij}\beta_M^{(j)}(0) + \sum_{j=1}^{N}\sum_{k=1}^{M} s_{ij}\left(\beta_M^{(j)}(1) - \beta_M^{(j)}(0)\right)P_j(t-k). \quad (11)$$

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Simulation Results

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- **For the** $\mathrm{IPCN}(M, N)$ **system:** we plot the average empirical sum at time t, which is given by

$$\frac{1}{N}\sum_{i=1}^{N} I_t(i) \quad \text{where} \quad I_t(i) = \frac{1}{t}\sum_{n=1}^{t} Z_{i,n}.$$

  For each plot, the average empirical sum is computed 100 times and the mean value is plotted against time.

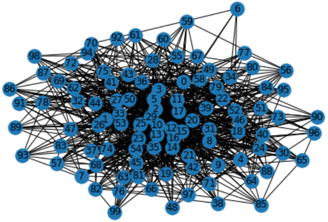
- **For the dynamical systems:** we plot the average infection rate at time t, which is given by

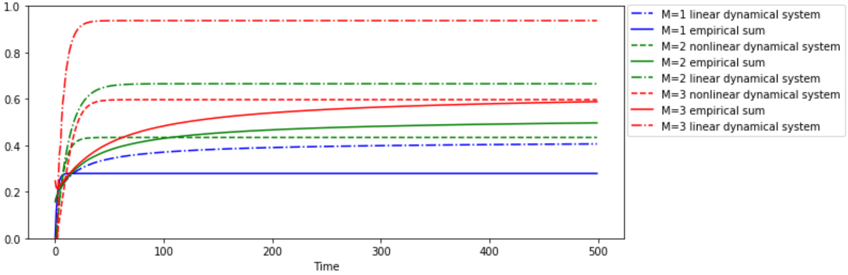
$$\frac{1}{N}\sum_{i=1}^{N} P_i(t).$$

Highly non-Homogeneous Interacting urns

A 100 node Barabási-Albert network

- *The nonlinear approximating dynamical system provides a good approximation and its quality increases with M.*
- *The linearization can perform poorly when the network is highly non-Homogeneous.*

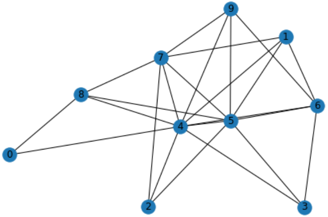


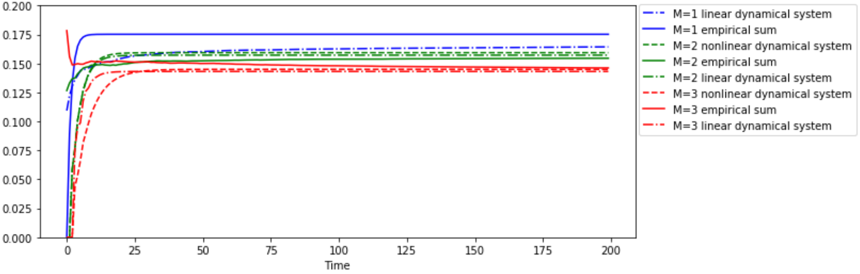


Nearly Homogeneous Interacting urns

A 10 node network

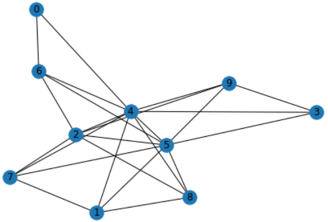
- *Similar to highly non-Homogeneous case, the nonlinear approximating dynamical system provides a good approximation and its quality increases with M.*
- *The linearization performs well when the network is near homogeneous.*

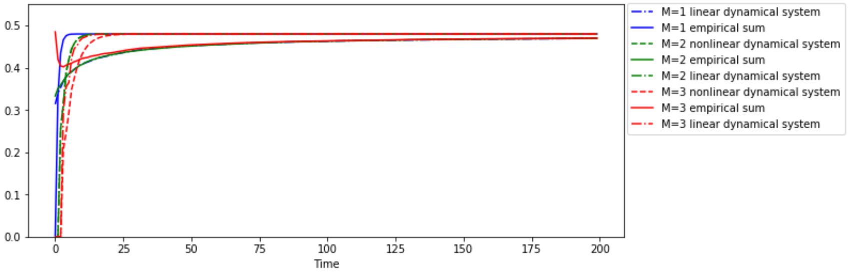




Homogeneous Interacting urns

For the Homogeneous case, the empirical sum and average infection rate (approximated using dynamical systems) converge to ρ irrespective of the memory of the system.





Conclusions and future directions

- A finite memory interacting Pólya urn network, $\mathrm{IPCN}(M, N)$
- Some stochastic properties $\mathrm{IPCN}(M, N)$
- Approximating dynamical systems for the $\mathrm{IPCN}(M, N)$

Future directions: Many avenues including

- Stability analysis for the approximating non-linear dynamical systems.
- Comparison of $\mathrm{IPCN}(M, N)$ with SIS and SIR models.

Thank You!