Almost Surely Convergence of Heterogeneous Deffuant-Weisbuch Model ⋆

Ge Chen [a], Wei Su [b], Wenjun Mei [c], Francesco Bullo [d],

[a] *National Center for Mathematics and Interdisciplinary Sciences & Key Laboratory of Systems and Control, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China*

[b] *School of Automation and Electrical Engineering, University of Science and Technology Beijing, Beijing 100083, China*

[c] *Automatic Control Laboratory, ETH, 8092 Zurich, Switzerland*

[d] *Department of Mechanical Engineering and the Center of Control, Dynamical-Systems and Computation, University of California at Santa Barbara, CA 93106-5070, USA*

Abstract

The Deffuant-Weisbuch (DW) model is a well-known opinion dynamics which attracts a lot of interest. However, this kind model is hard to analyze, even the most basic convergence of heterogeneous DW model is still open to prove. This paper solves the convergence problem on a basic heterogeneous DW model. It is shown that for any positive confidence bounds and initial states, the opinion of each agent will converge to a limit point almost surely. Also, the limit points of any two agents' opinions are same or have a distance larger than the confidence bounds of the two agents. Further more, some consensus conditions of the heterogeneous DW model are provided.

Key words: Opinion dynamics, consensus, Deffuant model, gossip model, bounded confidence model

1 Introduction

Opinion dynamics is a research field to study the dynamical processes of the formation, diffusion and evolution of public opinion about certain events and things in social systems. The study of opinion dynamics can trace back to the two-step flow of communication model (Katz and Lazarsfeld, 1955), and the social power model (French, 1956). Later, some new models and theories of opinion dynamics have been developed, such as the consensus model for group decision making (DeGroot, 1974), social influence network theory (Friedkin, 1998), social impact theory (Latané, 1981) and dynamic social impact theory (Latané, 1996). In recent years, significant attention has been attracted by bounded confidence (BC) models of opinion dynamics, which adopt a mechanism that one individual is not willing to accept the opinion of another one if he/she feels their opinions have a big gap. One well-known BC model is called as the Deffuant-Weisbuch (DW) model or Deffuant model (Deffuant et al., 2000). In this model a pair of agents is selected randomly at each time step, and each agent in the pair updates its opinion if the other agent's opinion in the pair is within its confidence bound. Another well-known BC model is called the Hegselmann-Krause (HK) model (Hegselmann and Krause, 2002), where all agents update their opinions synchronously by averaging the opinions in their confidence bounds.

The DW model was partly inspired by the famous Axelrod model about the dissemination of culture, and has been developed in a project about improving agri-environmental policies in the European union (Lorenz, 2007). There are several simulations for the DW model and some interesting phenomena have been found such as consensus, polarization and fragmentation (Lorenz, 2007, 2010). However, the DW model is hard to analyze, whose main difficulty lies in that the inter-agent topol-

⋆ This work was supported by the National Natural Science Foundation of China under grants 91427304, 61673373 and 11688101, the National Key Basic Research Program of China (973 program) under grant 2014CB845301/2/3, and the Leading research projects of Chinese Academy of Sciences under grant QYZDJ-SSW-JSC003.

Email addresses: chenge@amss.ac.cn (Ge Chen), suwei@amss.ac.cn (Wei Su), meiwenjunbd@gmail.com (Wenjun Mei), bullo@engineering.ucsb.edu (Francesco Bullo).

ogy is time-varying and coupled with the agents' states. Currently, the analysis of DW model focuses on the homogeneous case which means all the agents have a same confidence bound. The convergence of the homogeneous DW model has been proved in (Lorenz, 2005), while the convergence rate is provided in (Zhang and Chen, 2015). There are also some research on the modified DW models. For example, Como and Fagnani considered a generalized DW model with an interaction kernel and investigated its scaling limits when the number of the agents grows to infinite (Como and Fagnani, 2011); Zhang and Hong generalized the DW model by assuming that each agent can choose multi neighbors to exchange opinion at each time step (Zhang and Hong, 2013). However, the analysis of heterogeneous DW model is very few, even its convergence is still open to prove [1] (Lorenz, 2007). Similarly, the analysis of the HK model is also focused on the homogeneous case; however the convergence of the heterogeneous HK model is still lack of proof (MirTabatabaei and Bullo, 2012), except the special case that the confidence bound of each agent is either 0 or 1 (Chazelle and Wang, 2017).

This paper mainly proves the convergence of a basic heterogeneous DW model. It is shown that for any positive confidence bounds and initial states, the opinion of each agent will converge to a limit point almost surely. Also, the limit points of any two agents' opinions are same or have a distance larger than the confidence bounds of the two agents. Moreover, the consensus behavior of the DW model is studied.

The organization of this paper is as follows. Section 2 introduces the heterogeneous DW model and our main results, while Section 3 gives the proofs of our results. Section 4 provides the numerical simulations, and finally, Section 5 concludes the paper.

2 A basic heterogeneous DW model and main results

Following (Lorenz, 2007), this paper considers the following basic DW model: Assume there are $n(\geq 3)$ agents, and each agent i has an opinion value $x_i(t) \in \mathbb{R}$ at each time $t \geq 0$. Set $x(t) := (x_1(t), \ldots, x_n(t))$. Without loss of generality we assume $x(0) \in [0,1]^n$. Denote $r_i > 0$ as the confidence bound of the agent i. Without loss of generality, throughout this paper we assume

$$r_1 \geq r_2 \geq \cdots \geq r_n > 0.$$

Define $\mathbb{1}_{\{\cdot\}}$ to be the indicator function, i.e., $\mathbb{1}_{\{\omega\}} = 1$ if ω holds and $\mathbb{1}_{\{\omega\}} = 0$ otherwise. Let $\mathcal{V} = \{1, 2, \ldots, n\}$ be the index set of all agents. At each time $t \geq 0$, the DW model independently and uniformly selects a pair (i_t, j_t) for opinion updation from the set of all agents' pairs $\mathcal{A} = \{(i,j) : i, j \in \mathcal{V}, i \neq j\}$, and updates the opinions of the agents i_t and j_t by

$$\begin{cases} x_{i_t}(t+1) = x_{i_t}(t) \\ \qquad + \frac{1}{2}\mathbb{1}_{\{|x_{j_t}(t) - x_{i_t}(t)| \leq r_{i_t}\}}(x_{j_t}(t) - x_{i_t}(t)) \\ x_{j_t}(t+1) = x_{j_t}(t) \\ \qquad + \frac{1}{2}\mathbb{1}_{\{|x_{j_t}(t) - x_{i_t}(t)| \leq r_{j_t}\}}(x_{i_t}(t) - x_{j_t}(t)) \end{cases}. \qquad (1)$$

Meanwhile, the other agents' opinions remain unchanged, i.e.,

$$x_k(t+1) = x_k(t), \quad \forall k \in \mathcal{V} \setminus \{i_t, j_t\}. \qquad (2)$$

If $r_1 = r_2 = \cdots = r_n$ we call the model homogeneous, otherwise heterogeneous.

It has been shown that the homogeneous DW model (1)-(2) always converges to a limit opinion profile (Lorenz, 2005). Simulations show the convergence for the heterogeneous case, but the proof is lacking (Lorenz, 2007). We note that the original DW model (Deffuant et al., 2000) contains a weighting factor μ instead of $1/2$ in our protocol (1)-(2). Simulations show that the parameter μ mainly affect the convergence time (Deffuant et al., 2000; Weisbuch et al., 2002), so it was neglected by some previous studies to not further increase the complexity (Lorenz, 2007, 2010). Following previous work we also do not consider the parameter μ here.

Before the statement of our results, we need to define the probability space of the DW model. If the initial state is deterministic, we let $\Omega = \mathcal{A}^\infty$ be the sample space, $\mathcal{F}$ be the Borel σ-algebra of Ω, and P be the probability measure on $\mathcal{F}$. Then, the probability space of the DW model is written as $(\Omega, \mathcal{F}, P)$. If the initial state is stochastic, we let $\Omega = [0,1]^n \times \mathcal{A}^\infty$ be the sample space, and similar to the case of deterministic initial state the probability space is defined by $(\Omega, \mathcal{F}, P)$.

Theorem 1 (Convergence of heterogeneous DW model) *Consider the heterogeneous DW model (1)-(2) with positive confidence bounds. Then for any initial state $x(0) \in [0,1]^n$, there exists a random vector $x^* \in [0,1]^n$ satisfying $x_i^* = x_j^*$ or $|x_i^* - x_j^*| > \max\{r_i, r_j\}$ for all $i \neq j$, such that $x(t)$ converges to x^* almost surely, i.e.,*

$$P\left(\omega \in \Omega : \lim_{t\to\infty} x(t)(\omega) = x^*(\omega)\right) = 1.$$

The proof of Theorem 1 is put in Section 3. Also, from this theorem we can get the following two corollaries on consensus, which means that all agents' opinions converge to a same value.

[1] Zhang and Hong claimed the proof of the convergence of heterogeneous DW model in the 31st Chinese Control Conference, 2012:1124-1129; however the proof turns out to be incorrect later.

Corollary 2 *Consider the heterogeneous DW model (1)-(2) with positive confidence bounds. If* $r_1 \geq 1$*, then for any initial state* $x(0) \in [0,1]^n$ *the system reaches consensus almost surely.*

Corollary 3 *Consider the heterogeneous DW model (1)-(2) with positive confidence bounds. Assume the initial state* $x(0)$ *is randomly distributed in* $[0,1]^n$ *whose joint probability density has a lower bound* $\underline{\rho} > 0$*, i.e., for any real numbers* a_i, b_i *with* $0 \leq a_i < b_i \leq 1$*,* $1 \leq i \leq n$*,*

$$P\Big(\bigcap_{i=1}^{n}\{x_i(0) \in [a_i, b_i]\}\Big) \geq \underline{\rho}\prod_{i=1}^{n}(b_i - a_i). \tag{3}$$

Then, the system reaches consensus almost surely if and only if $r_1 \geq 1$*.*

The proofs of Corollaries 2 and 3 are put in Section 3.

3 Proof of Main Results

The proof of Theorem 1 is very complex. We adopt the method "transforming the analysis of a stochastic system into the design of control algorithms" first proposed by (Chen, 2017). This method need construct a new system called as DW-control system to help the analysis of the DW model.

3.1 DW-control system and connection to DW model

In the DW protocol (1)-(2), if at each time t the pair (i_t, j_t) for opinion updation is not selected randomly but treated as a control input, which means that (i_t, j_t) can be chosen from the set $\mathcal{A}$ arbitrarily, we call such a system as the *DW-control system.*

Given $S \subseteq \mathbb{R}^n$, we say S *is reached at time* t if $x(t) \in S$, and *is reached in the time interval* $[t_1, t_2]$ if there exists $t \in [t_1, t_2]$ such that $x(t) \in S$.

Definition 4 *Let* $S_1, S_2 \subseteq [0,1]^n$ *be two state sets. Under the DW-control system,* S_1 *is said to be finite-time reachable from* S_2 *if: There exists an integer* $t^* > 0$ *such that for any* $x(0) \in S_2$*, we can find a sequence of pairs* $(i_0', j_0'), (i_1', j_1'), \ldots, (i_{t^*-1}', j_{t^*-1}')$ *for opinion updation which guarantees* S_1 *is reached in the time interval* $[0, t^*]$*.*

Based on these definitions we can get the following lemma:

Lemma 5 (Connection between DW model and DW-control system) *Let* $S \subseteq [0,1]^n$ *be a set of states. Assume* S *is finite-time reachable from* $[0,1]^n$ *under the DW-control system. Then, under the DW protocol, for any initial state* $x(0) \in [0,1]^n$*, with probability* 1 *there exists a finite time* t *such that* S *is reached in* $[0,t]$*.*

Proof. First according to the rule of the DW protocol (1)-(2) we get $x(t) \in [0,1]^n$ for all $t \geq 0$. Also, since S is reached in finite time from $[0,1]^n$ under the DW-control system, by Definition 4 there exist an integer t^* such that for any $x(0) \in [0,1]^n$, we can find a sequence of pairs $(i_0', j_0'), (i_1', j_1'), \ldots, (i_{t^*-1}', j_{t^*-1}')$ which guarantees S is reached in $[0, t^*]$. From this and the definition of the DW-control system, for any $t \geq 0$ and $x(t) \in [0,1]^n$, there exists a sequence of pairs $(i_t', j_t'), (i_{t+1}', j_{t+1}'), \ldots, (i_{t+t^*-1}', j_{t+t^*-1}')$ such that S is reached in $[t, t+t^*]$. Thus, under the DW protocol, for any $t \geq 0$ and $x(t) \in [0,1]^n$ we have

$$\begin{aligned}
&P\left(\{S \text{ is reached in } [t, t+t^*]\}\,|x(t)\right) \\
&\geq P\Big(\bigcap_{s=t}^{t+t^*-1}\big\{(i_s, j_s) = (i_s', j_s')\big\}|x(t)\Big) \\
&= P\Big((i_t, j_t) = (i_t', j_t')|x(t)\Big) \\
&\cdot P\Big(\bigcap_{s=t+1}^{t+t^*-1}\big\{(i_s, j_s) = (i_s', j_s')\big\}|x(t), (i_t, j_t) = (i_t', j_t')\Big) \\
&= \cdots = P\Big((i_t, j_t) = (i_t', j_t')|x(t)\Big) \\
&\cdot P\Big(\big\{(i_{t+1}, j_{t+1}) = (i_{t+1}', j_{t+1}')\big\}|x(t), (i_t, j_t) = (i_t', j_t')\Big) \\
&\cdots P\Big(\big\{(i_{t+t^*-1}, j_{t+t^*-1}) = (i_{t+t^*-1}', j_{t+t^*-1}')\big\} \\
&\qquad |x(t), (i_s, j_s) = (i_s', j_s'), s \in [t, t+t^*-2]\Big),
\end{aligned} \tag{4}$$

where all the equations use Bayes' theorem. Because (i_t, j_t) is uniformly and independently selected from the set A, for any $x(s) \in [0,1]^n$ we get

$$P\big\{(i_s, j_s) = (i_s', j_s')|x(s)\big\} = \frac{1}{|A|} = \frac{1}{n(n-1)}, \tag{5}$$

where $|A|$ denotes the cardinality of the set A. Substituting (5) into (4) yields

$$\begin{aligned}
&P\left(\{S \text{ is reached in } [t, t+t^*]\}\,|x(t)\right) \\
&\geq \frac{1}{n^{t^*}(n-1)^{t^*}}.
\end{aligned} \tag{6}$$

Set E_t to be the event that S is reached in $[t, t+t^*]$, and let E_t^c be the complement set of E_t. For any integer $M > 0$ and $x(0) \in [0,1]^n$, using Bayes' theorem again and (6) we have

$$P\big(\{S \text{ is not reached in } [0,(t^*+1)M-1]\}\,|x(0)\big)$$
$$= P\Big(\bigcap_{m=0}^{M-1} E^c_{m(t^*+1)}|x(0)\Big)$$
$$= P\left(E^c_0|x(0)\right)\prod_{m=1}^{M-1} P\Big(E^c_{m(t^*+1)}|x(0),\bigcap_{0\le m'<m} E^c_{m'(t^*+1)}\Big)$$
$$\le \Big(1-\frac{1}{n^{t^*}(n-1)^{t^*}}\Big)^M.$$

Let M grow to infinite we get

$$P\big(\{S \text{ is not reached in } [0,\infty)\}\,|x(0)\big)=0,$$

which indicates that with probability 1 there exists a finite time t such that S is reached in $[0,t]$. □

According to Lemma 5, to prove the convergence of the DW model, we only need design control algorithms for DW-control system such that a convergence set is reached. Before the design of control algorithms we introduce an important definition named as maximal-confidence cluster in the following subsection.

3.2 Maximal-confidence clusters and properties

Recall that we assume $r_1\ge r_2\ge\cdots\ge r_n>0$. For any opinion state $x=(x_1,\ldots,x_n)\in[0,1]^n$, let $C_1(x)\subseteq\mathcal{V}$ be the cluster of the agents that can connect to agent 1 directly or indirectly with the confidence bound r_1, i.e., $i\in C_1(x)$ if and only if $|x_i-x_1|\le r_1$ or there exists some agents $1',2',\ldots,k'\in\mathcal{V}$ such that $|x_i-x_{1'}|\le r_1, |x_{1'}-x_{2'}|\le r_1,\ldots,|x_{k'}-x_1|\le r_1$. From this definition we have $1\in C_1(x)$.

Set $\widetilde{C}_1(x):=\mathcal{V}\setminus C_1(x)$. If $\widetilde{C}_1(x)$ is not empty, we let $i_2:=\min_{i\in\widetilde{C}_1(x)} i$ and define $C_2(x)\subseteq\widetilde{C}_1(x)$ to be the cluster of all agents that can connect to agent i_2 directly or indirectly with the confidence bound r_{i_2}. Set $\widetilde{C}_2(x):=\mathcal{V}\setminus C_2(x)$. If $\widetilde{C}_2(x)$ is not empty, we let $i_3:=\min_{i\in\widetilde{C}_2(x)} i$ and define $C_3(x)\subseteq\widetilde{C}_2(x)$ to be the cluster of all agents that can connect to agent i_3 directly or indirectly with the confidence bound r_{i_3}. Repeat this process until each agent is divided into one cluster. We call the clusters $C_1(x),C_2(x),\ldots,C_K(x)$ as the *maximal-confidence clusters*.

Two maximal-confidence clusters are shown in Fig. 1. Also, we give the following lemma to describe the distance between maximal-confidence clusters:

Lemma 6 (Distance between maximal-confidence clusters) *For any opinion state $x\in[0,1]^n$ and two*

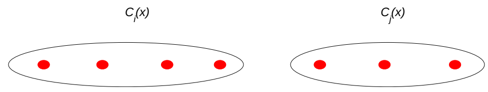

Fig. 1. Two maximal-confidence clusters $C_i(x)$ and $C_j(x)$. The distance between two adjacent nodes in $C_i(x)$ (or $C_j(x)$) is not bigger than the maximal confidence bound of $C_i(x)$ (or $C_j(x)$), while the distance between these two clusters is bigger than the maximal confidence bound of $C_i(x)$ and $C_j(x)$.

different maximal-confidence clusters $C_i(x)$ and $C_j(x)$, let $r^{ij}_{\max}:=\max_{k\in C_i(x)\cup C_j(x)} r_k$ be the maximal confidence bound of all agents in $C_i(x)$ and $C_j(x)$. Then, the opinion values of agents in $C_i(x)$ are all $r^{ij}_{\max}$ bigger or smaller than those in $C_j(x)$, i.e.,

$$x_k-x_l>r^{ij}_{\max}\quad\forall k\in C_i(x),l\in C_j(x),$$

or

$$x_l-x_k>r^{ij}_{\max}\quad\forall k\in C_i(x),l\in C_j(x).$$

Proof. Without loss of generality we assume that $\max_{k\in C_i(x)} r_k=r^{ij}_{\max}$. Let $x^i_{\min}:=\min_{k\in C_i(x)} x_k$ and $x^i_{\max}:=\max_{k\in C_i(x)} x_k$ denote the minimal and maximal opinion values of all agents in $C_i(x)$ respectively. For any $l\in C_j(x)$, if $x_l\in[x^i_{\min}-r^{ij}_{\max},x^i_{\max}+r^{ij}_{\max}]$, by the definition of the maximal-confidence cluster we have $l\in C_i(x)$, which is contradictory with $l\in C_j(x)$. Thus, for any $l\in C_j(x)$, we get

$$x_l<x^i_{\min}-r^{ij}_{\max}\le x_k-r^{ij}_{\max},\quad\forall k\in C_i(x)\qquad(7)$$

or

$$x_l>x^i_{\max}+r^{ij}_{\max}\ge x_k+r^{ij}_{\max},\quad\forall k\in C_i(x).\qquad(8)$$

Since $C_j(x)$ is also a maximal-confidence cluster, there is no agent in $C_i(x)$ whose opinion value is located in the interval $[x^j_{\min},x^j_{\max}]$. Thus, either (7) or (8) holds for all $l\in C_j(x)$. □

Under the DW protocol (1)-(2), the maximal-confidence clusters have the convex property as follows:

Lemma 7 (Convexity of maximal-confidence clusters) *Consider the DW protocol (1)-(2) with arbitrary initial state and updation pairs $\{(i_t,j_t)\}_{t\ge0}$. For any $t\ge0$ and any maximal-confidence cluster $C_i(x(t))$, the opinion values of all agents in $C_i(x(t))$ will always stay in the interval $[x^i_{\min}(t),x^i_{\max}(t)]$ at the time $s\ge t$, i.e.,*

$$x^i_{\min}(t)\le x_j(s)\le x^i_{\max}(t),\quad\forall j\in C_i(x(t)),s\ge t,$$

where $x^i_{\min}(t) := \min_{k\in C_i(x(t))} x_k(t)$ *and* $x^i_{\max}(t) := \max_{k\in C_i(x(t))} x_k(t)$ *denote the minimal and maximal opinion values of all agents in* $C_i(x(t))$ *respectively.*

Proof. Assume that at time t all maximal-confidence clusters are $C_1 = C_1(x(t)), C_2 = C_2(x(t)), \ldots, C_K = C_K(x(t))$. By Lemma 6 we can order these clusters as

$$C_{j_1} \prec C_{j_2} \prec \cdots \prec C_{j_K},$$

and get

$$\begin{aligned} &\min_{l\in C_{j_{k+1}}} x_l(t) - \max_{l\in C_{j_k}} x_l(t) \\ &> r^{k,k+1}, \quad \forall 1 \le k \le K-1, \end{aligned} \tag{9}$$

where $C_i \prec C_j$ means that at time t the opinion values of the agents in C_i are all less than those in C_j, and $r^{k,k+1} := \max_{l\in C_{j_k}\cup C_{j_{k+1}}} r_l$.

By the DW protocol (1)-(2), if the updation pair (i_t, j_t) belongs to different maximal-confidence clusters then from (9) we have $x_{i_t}(t+1) = x_{i_t}(t)$ and $x_{j_t}(t+1) = x_{j_t}(t)$; if (i_t, j_t) belongs to a same maximal-confidence cluster C_{j_k} then $x_{i_t}(t+1)$ and $x_{j_t}(t+1)$ will stay in the interval $[x^{j_k}_{\min}(t), x^{j_k}_{\max}(t)]$. Thus,

$$\begin{aligned} &\min_{l\in C_{j_{k+1}}} x_l(t+1) - \max_{l\in C_{j_k}} x_l(t+1) \\ &> r^{k,k+1}, \quad \forall 1 \le k \le K-1. \end{aligned}$$

Repeating this process yields our result. □

With the definition and properties of maximal-confidence clusters we can design control algorithms and complete final proof of our results in the following subsection.

3.3 Design of control algorithms and final proofs

For any opinion state $x \in [0,1]^n$ and any maximal-confidence cluster $C_i(x)$, we say $C_i(x)$ is a *complete cluster* if any agent in $C_i(x)$ can interact with others with the minimal confidence bound of $C_i(x)$, i.e.,

$$\max_{j,k\in C_i(x)} |x_j - x_k| \le \min_{j\in C_i(x)} r_j.$$

Lemma 8 *Let* $t \ge 0$ *and* $x(t) \in [0,1]^n$ *be arbitrarily given. Let* $C_i(x(t))$ *be an arbitrary maximal-confidence cluster, in which the agents' maximal and minimal confidence bounds are* $r^i_{\max}$ *and* $r^i_{\min}$ *respectively. Assume*

$$\max_{M,m\in C_i(x(t))} [x_M(t) - x_m(t)] > r^i_{\min}. \tag{10}$$

Then, under the DW-control system, there is a sequence of agent pairs $(i'_t, j'_t), (i'_{t+1}, j'_{t+1}), \ldots, (i'_{t+t^*-1}, j'_{t+t^*-1})$ *with*

$$t^* \le (|C_i(x(t))| - 1)^2 \left(1 + \lceil \log_2 \lceil r^i_{\max}/r^i_{\min} \rceil \rceil\right)$$

for opinion updation, such that one of the following two results holds:
i) the agents in $C_i(x(t))$ *split into different maximal-confidence clusters at time* $t + t^*$*;*

ii) $$\begin{aligned} &\max_{M,m\in C_i(x(t))} [x_M(t+t^*) - x_m(t+t^*)] \\ &\le \max_{M,m\in C_i(x(t))} [x_M(t) - x_m(t)] - r^i_{\min}/4. \end{aligned}$$

The proof of Lemma 8 is quite complex put in Appendix A.

Lemma 9 *Let* $t \ge 0$ *and* $x(t) \in [0,1]^n$ *be arbitrarily given. Let* $C_i(x(t))$ *be an arbitrary maximal-confidence cluster. Assume*

$$\max_{M,m\in C_i(x(t))} [x_M(t) - x_m(t)] \le \min_{m\in C_i(x(t))} r_m. \tag{11}$$

Then, under the DW-control system, there is a sequence of agent pairs $(i'_t, j'_t), (i'_{t+1}, j'_{t+1}), \ldots, (i'_{t+t^*-1}, j'_{t+t^*-1})$ *with* $t^* \le |C_i(x(t))|/2$ *for opinion updation, such that*

$$\begin{aligned} &\max_{M,m\in C_i(x(t))} [x_M(t+t^*) - x_m(t+t^*)] \\ &\le \frac{2}{3} \max_{M,m\in C_i(x(t))} [x_M(t) - x_m(t)]. \end{aligned}$$

Proof. The proof of this lemma is similar for the cases $t = 0, 1, 2, \ldots$. To simplify the exposition we only consider the case when $t = 0$. We set

$$\begin{aligned} x^i_{\min}(0) &= \min_{m\in C_i(x(0))} x_m(0), \\ x^i_{\max}(0) &= \max_{m\in C_i(x(0))} x_m(0). \end{aligned}$$

Let $d(0) := x^i_{\max}(0) - x^i_{\min}(0)$. Set

$$\underline{B}(s) := \{m \in C_i(x(0)) : x_m(s) < x^i_{\min}(0) + d(0)/3\}.$$

and

$$\overline{B}(s) := \{m \in C_i(x(0)) : x_m(s) > x^i_{\max}(0) - d(0)/3\}.$$

Take $I = |\underline{B}(0)|$ and $J = |\overline{B}(0)|$. Without loss of generality we assume $I \le J$.

Label the elements in $\underline{B}(0)$ as $i_0, i_1, \ldots, i_{I-1}$, and the elements in $\overline{B}(0)$ as $j_0, j_1, \ldots, j_{J-1}$. For $0 \le k \le I-1$, we choose (i_k, j_k) as the agent pair for opinion updation at time k, then by the protocol (1)-(2) and (11) we get

$$\begin{aligned} x_{i_k}(I) &= x_{j_k}(I) = x_{i_k}(k+1) \\ &= \frac{x_{i_k}(k) + x_{j_k}(k)}{2} = \frac{x_{i_k}(0) + x_{j_k}(0)}{2} \\ &\in \Big(\frac{x^i_{\min}(0) + x^i_{\max}(0) - \frac{d(0)}{3}}{2}, \\ &\quad \frac{x^i_{\min}(0) + \frac{d(0)}{3} + x^i_{\max}(0)}{2} \Big) \\ &= \Big(x^i_{\min}(0) + \frac{d(0)}{3}, x^i_{\max}(0) - \frac{d(0)}{3} \Big), \end{aligned}$$

which indicates $\underline{B}(I) = \emptyset$. Combining this with Lemma 7 we have

$$\begin{aligned} &\max_{M,m \in C_i(x(t))} [x_M(I) - x_m(I)] \\ &\le x^i_{\max}(0) - x^i_{\min}(0) - \frac{d(0)}{3} = \frac{2d(0)}{3}. \end{aligned}$$

Also, because $I \le |C_i(x(t))|/2$, our result is obtained. □

For any opinion state $x = (x_1, \ldots, x_n) \in [0,1]^n$, let $D(x)$ denote the maximal diameter among all the maximal-confidence clusters $C_1(x), C_2(x), \ldots, C_K(x)$, i.e.,

$$D(x) = \max_{1 \le i \le K} \max_{j,k \in C_i(x)} |x_j - x_k|. \tag{12}$$

Lemma 10 *Consider the DW-control system. Let $r_{\min}$ and $r_{\max}$ be the maximal and minimal confidence bounds of all agents. Then for any initial state and constant $\varepsilon > 0$, there exists a sequence of agent pairs $(i'_0, j'_0), (i'_1, j'_1), \ldots, (i'_{t^*-1}, j'_{t^*-1})$ with*

$$\begin{aligned} t^* \le (n-1)^2 &\left(1 + \left\lceil \log_2 \left\lceil \frac{r_{\max}}{r_{\min}} \right\rceil \right\rceil \right) \left\lceil \frac{4(1-r_{\min})}{r_{\min}} \right\rceil \\ &+ \frac{n}{2} \left\lceil \frac{-\log \varepsilon}{\log 3/2} \right\rceil. \end{aligned}$$

for opinion updation such that $D(x(t^*)) \le \varepsilon$.

Proof. Assume there are K_t maximal-confidence clusters $C_1(x(t)), C_2(x(t)), \ldots, C_{K_t}(x(t))$ at time t. Using Lemma 8 repeatedly there exists a sequence of agent pairs $(i'_0, j'_0), (i'_1, j'_1), \ldots, (i'_{T_1-1}, j'_{T_1-1})$ for opinion updation such that

$$\max_{M,m \in C_i(x(T_1))} [x_M(T_1) - x_m(T_1)] \le \min_{m \in C_i(x(T_1))} r_m$$

for all $1 \le i \le K_{T_1}$. Since $(c_1-1)^2 + \cdots + (c_m-1)^2 \le (c_1 + \cdots + c_m - 1)^2$ for any positive integers $m, c_1, \ldots, c_m$, by Lemma 8 it can be computed that

$$\begin{aligned} T_1 \le (n-1)^2 & \\ \times \left(1 + \left\lceil \log_2 \left\lceil \frac{r_{\max}}{r_{\min}} \right\rceil \right\rceil \right) &\left\lceil \frac{4(1-r_{\min})}{r_{\min}} \right\rceil. \end{aligned}$$

Further, using Lemma 9 repeatedly there exists a sequence of agent pairs $(i'_{T_1}, j'_{T_1}), (i'_{T_1+1}, j'_{T_1+1}), \ldots, (i'_{T_1+T_2-1}, j'_{T_1+T_2-1})$ for opinion updation such that

$$D(x(T_1 + T_2)) \le \varepsilon.$$

By Lemma 9, it can be computed that $T_2 \le \frac{n}{2} \lceil \frac{-\log \varepsilon}{\log 3/2} \rceil$.

□

Proof of Theorem 1. For any constant $\varepsilon > 0$, let S_ε be the state set defined by

$$S_\varepsilon := \{x \in [0,1]^n : D(x) \le \varepsilon\},$$

where $D(x)$ is the maximal diameter of all maximal-confidence clusters defined by (12). By Lemma 10, S_ε is finite-time reachable from $[0,1]^n$ under the DW-control system. So, under the DW protocol (1)-(2), by Lemma 5 almost surely there exists a finite time $t_\varepsilon > 0$ such that S_ε is reached in $[0, t_\varepsilon]$ for any initial state. By the convexity of maximal-confidence clusters (Lemma 7) we have $D(x(t)) \le \varepsilon$ for all $t \ge t_\varepsilon$. Let $\varepsilon \to 0^+$ we can get

$$P\left(\omega \in \Omega : \lim_{t \to \infty} D(x(t))(\omega) = 0\right) = 1.$$

From this and Lemma 7 we have that $x(t)$ almost surely converges to a random vector x^*. By Lemma 6 we obtain $x^*_i = x^*_j$ or $|x^*_i - x^*_j| > \max\{r_i, r_j\}$ for any $i \ne j$. □

Proof of Corollary 2. By Theorem 1 we have $x(t)$ almost surely converges to a limit point $x^* \in [0,1]^n$ which satisfies either $|x^*_1 - x^*_i| = 0$ or $|x^*_1 - x^*_i| > r_1$ for all $2 \le i \le n$. Because $r_1 \ge 1$, we have $|x^*_1 - x^*_i| = 0$ for all $2 \le i \le n$, which indicates x^* is a consensus state. □

Proof of Corollary 3. If $r_1 \ge 1$, by Corollary 2 the system reaches consensus almost surely.

If $r_1 < 1$, by (3) we have

$$\begin{aligned} &P\Big(x_1(0) \in \Big[0, \frac{1-r_1}{3}\Big], \bigcap_{i=2}^n \Big\{x_i(0) \in \Big[\frac{2+r_1}{3}, 1\Big]\Big\}\Big) \\ &\ge \underline{\rho} \left(\frac{1-r_1}{3}\right)^n. \end{aligned}$$

Also, if $x_1(0) \in [0, \frac{1-r_1}{3}]$ and $\bigcap_{i=2}^n \{x_i(0) \in [\frac{2+r_1}{3}, 1]\}$ happens, we have $|x_1(0) - x_i(0)| = \frac{1+2r_1}{3} > r_1$ for

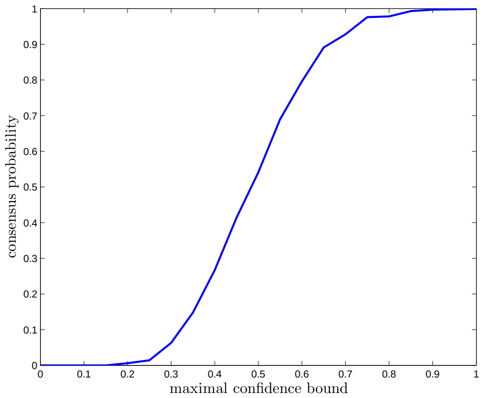

Fig. 2. The consensus probability with respect to the maximal confidence bound.

$2 \leq i \leq n$, which results in the system cannot reach consensus because the agent 1 can never interact with the agents $2, \ldots, n$. □

4 Simulations

Corollary 3 provides a sufficient and necessary condition for almost surely consensus when the initial opinions are randomly distributed. However, the exact value of consensus probability is unknown. In this section, we provide some simulations for the consensus probability of the heterogeneous DW model. Let $n = 10$. Assume the initial opinions are independently and uniformly distributed on $[0, 1]$. Suppose the agent 1 has a maximal confidence bound $r_{\max}$, while the confidence bounds of other agents are independently and uniformly distributed on $[0, r_{\max}]$. Fig. 2 shows the consensus probability of the heterogeneous DW model (1)-(2) with respect to the maximal confidence bound $r_{\max}$.

5 Conclusions

Bounded confidence (BC) models of opinion dynamics, which adopt a mechanism that one individual is not willing to accept the opinion of another one if he/she feels their opinions have a big gap, have attracted significant attention in recent years. One well-known BC model is called as the Deffuant-Weisbuch (DW) model, in which a pair of agents is selected randomly at each time step, and each agent in the pair updates its opinion if the other agent's opinion in the pair is within its confidence bound. Because the inter-agent topology of the DW model is time-varying and coupled with the agents' states, the heterogeneous DW model is hard to analyze. This paper proves the convergence of a basic heterogeneous DW model, while the properties of the convergent points are studied.

It remains to prove the convergence of the original heterogeneous DW model which contains a weighting factor μ instead of $1/2$ in our protocol (1)-(2). The original heterogeneous DW model has a higher complexity, and we need design more ingenious control algorithms such that the DW-control system converges to a set with invariant topology in finite time.

References

B. Chazelle and C. Wang. Inertial hegselmann-krause systems. *IEEE Transactions on Automatic Control*, 62(8):3905–3913, 2017.

G. Chen. Small noise may diversify collective motion in vicsek model. *IEEE Transactions on Automatic Control*, 62(2):636–651, 2017.

G. Como and F. Fagnani. Scaling limits for continuous opinion dynamics systems. *The Annals of Applied Probability*, 21(4):1537–1567, 2011.

G. Deffuant, D. Neau, F. Amblard, and G. Weisbuch. Mixing beliefs among interacting agents. *Advances in Complex Systems*, 3(1/4):87–98, 2000.

M. H. DeGroot. Reaching a consensus. *Journal of the American Statistical Association*, 69(345):118–121, 1974. doi: 10.1080/01621459.1974.10480137.

J. R. P. French. A formal theory of social power. *Psychological Review*, 63(3):181–194, 1956. doi: 10.1037/h0046123.

N. E. Friedkin. *A Structural Theory of Social Influence*. Structural Analysis in the Social Sciences. Cambridge University Press, 1998. ISBN 9780521454827.

R. Hegselmann and U. Krause. Opinion dynamics and bounded confidence models, analysis, and simulations. *Journal of Artificial Societies and Social Simulation*, 5(3), 2002. URL `http://jasss.soc.surrey.ac.uk/5/3/2.html`.

E. Katz and F. Lazarsfeld. *Personal influence: the part played by people in the flow of mass communications*. Free Press, Glencoe, IL, 1955.

B. Latané. The psychology of social impact. *American Psychologist*, 36(4):343–365, 1981. doi: 10.1037/0003-066X.36.4.343.

B. Latané. Dynamic social impact: The creation of culture by communication. *Journal of Communication*, 46(4):13–25, 1996. doi: 10.1111/j.1460-2466.1996.tb01501.x.

J. Lorenz. A stabilization theorem for dynamics of continuous opinions. *Physica A: Statistical Mechanics and its Applications*, 355(1):217–223, 2005.

J. Lorenz. Continuous opinion dynamics under bounded confidence: A survey. *International Journal of Modern Physics C*, 18(12):1819–1838, 2007.

J. Lorenz. Heterogeneous bounds of confidence: Meet, discuss and find consensus! *Complexity*, 4(15):43–52, 2010.

A. MirTabatabaei and F. Bullo. Opinion dynamics in heterogeneous networks: Convergence conjectures and theorems. 50(5):2763–2785, 2012. doi: 10.1137/11082751X.

G. Weisbuch, G. Deffuant, F. Amblard, and J. P. Nadal.

Meet, discuss, and segregate! *Complexity*, 7(3):55–63, 2002.

J. B. Zhang and G. Chen. Convergence rate of the asymmetric deffuant-weisbuch dynamics. *Journal of Systems Science and Complexity*, 28(4):773–787, 2015.

J. B. Zhang and Y. G. Hong. Opinion evolution analysis for short-range and long-range deffuantcweisbuch models. *Physica A: Statistical Mechanics and Its Applications*, 392(21):5289–5297, 2013.

A The proof of Lemma 8

The proof of this lemma is completely similar for all cases $t = 0, 1, 2, \ldots$. To simplify the exposition we only consider the case when $t = 0$.

Assume the agents j and k have the minimal and maximal opinions among $C_i(x(0))$ at time 0 respectively, i.e.,

$$x_j(0) = \min_{m \in C_i(x(0))} x_m(0), \quad x_k(0) = \max_{m \in C_i(x(0))} x_m(0).$$

Also, assume that the agent l has the maximal confidence bound $r^i_{\max}$ in $C_i(x(0))$.

We first consider the case when $x_l(0) \geq \frac{x_k(0)+x_j(0)}{2}$. From (10) we have

$$x_l(0) \geq x_j(0) + r^i_{\min}/2. \tag{A.1}$$

Let

$$\underline{A}(s) := \{m \in C_i(x(0)) : x_m(s) < x_j(0) + r^i_{\min}/4\}.$$

We aims to control the agent pairs for opinion updation such that $\underline{A}(s)$ becomes empty in finite time. The control strategy can be divided into the following steps:

Step 1: Control the agent pairs for opinion updation until one of the following two events happens:
(E1) The agents in $C_i(x(0))$ split into different maximal-confidence clusters;
(E2) $|\underline{A}(s)| = |\underline{A}(0)| - 1$, where $|\cdot|$ denote the cardinality of a set.

Let i'_0 be the agent in $C_i(x(0))$ which has the smallest opinion within the confidence bound of agent l at time 0, i.e.,

$$i'_0 = \underset{m \in C_i(x(0))}{\arg\min} \{x_m(0) : |x_l(0) - x_m(0)| \leq r_l\}.$$

We continue our discussion by considering the following two cases:
Case I: $i'_0 \in \underline{A}(0)$. Choose (i'_0, l) as the agent pair for opinion updation at times $0, 1, \ldots, \lceil \log_2 \lceil \frac{x_l(0) - x_{i'_0}(0)}{r_{i'_0}} \rceil \rceil :=$ T. We can get $T \leq \lceil \log_2 \lceil n r_l / r_{i'_0} \rceil \rceil$ is uniformly bounded, and

$$\begin{aligned} &T - 1 < \log_2 \lceil \frac{x_l(0) - x_{i'_0}(0)}{r_{i'_0}} \rceil \leq T \iff \\ &2^{T-1} < \lceil \frac{x_l(0) - x_{i'_0}(0)}{r_{i'_0}} \rceil \leq 2^T \iff \\ &2^{T-1} r_{i'_0} < x_l(0) - x_{i'_0}(0) \leq 2^T r_{i'_0}. \end{aligned} \tag{A.2}$$

If $T = 0$, by (A.2) we have $x_l(0) - x_{i'_0}(0) \leq r_{i'_0}$, then by the protocol (1)-(2) and (A.1) we get

$$\begin{aligned} x_{i'_0}(1) = x_l(1) &= \frac{x_l(0) + x_{i'_0}(0)}{2} \\ &\geq \frac{x_l(0) + x_j(0)}{2} \geq x_j(0) + r^i_{\min}/4, \end{aligned}$$

which implies $|\underline{A}(1)| = |\underline{A}(0)| - 1$.

If $T \geq 1$, by (A.2) and using the protocol (1)-(2) repeatedly we can computer that

$$\begin{cases} x_{i'_0}(s) = x_{i'_0}(0) \\ x_l(s) = x_{i'_0}(0) + \frac{1}{2^s}(x_l(0) - x_{i'_0}(0)) \end{cases}, \quad s = 1, \ldots, T,$$

and

$$\begin{aligned} &x_{i'_0}(T+1) \\ &= x_l(T+1) \\ &= x_{i'_0}(0) + \frac{1}{2^{T+1}}(x_l(0) - x_{i'_0}(0)) \\ &> x_{i'_0}(0) + r_{i'_0}/4 \geq x_{i'_0}(0) + r^i_{\min}/4 \\ &\geq x_j(0) + r^i_{\min}/4, \end{aligned} \tag{A.3}$$

which implies $|\underline{A}(T+1)| = |\underline{A}(0)| - 1$.
Case II: $i'_0 \notin \underline{A}(0)$. Choose (i'_0, l) as the agent pair for opinion updation at times $0, 1, \ldots, \lceil \log_2 \lceil \frac{x_l(0) - x_{i'_0}(0)}{r_{i'_0}} \rceil \rceil :=$ T_1. Similar to (A.3) we get

$$\begin{aligned} x_l(T_1+1) &= x_{i'_0}(T_1+1) \\ &= x_{i'_0}(0) + \frac{1}{2^{T_1+1}}(x_l(0) - x_{i'_0}(0)) \\ &< x_l(0). \end{aligned} \tag{A.4}$$

Let $\mathcal{L}_l(s)$ denote the set of the agents in $C_i(x(0))$ whose opinions at time s are less than $x_l(s)$, i.e.,

$$\mathcal{L}_l(s) := \{m \in C_i(x(0)) : x_m(s) < x_l(s)\}.$$

By (A.4) and with the fact that all agents except l and i'_0 keep their opinions invariant during the time $[0, T_1+1]$, we have

$$|\mathcal{L}_l(T_1+1)| \le |\mathcal{L}_l(0)| - 1. \tag{A.5}$$

Let i_1' be the agent in $C_i(x(0))$ which has the smallest opinion within the confidence bound of agent l at time T_1+1, i.e.,

$$i_1' = \arg\min_{m\in C_i(x(0))} \{x_m(T_1+1) : \\ |x_l(T_1+1) - x_m(T_1+1)| \le r_l\}.$$

If $x_{i_1'}(T_1+1) = x_l(T_1+1)$, the agents in $C_i(x(0))$ split into different maximal-confidence clusters; otherwise, choose (i_1', l) as the agent pair for opinion updation at times $T_1+1, T_1+2, \ldots, T_1+1+\lceil\log_2\lceil\frac{x_l(T_1+1)-x_{i_1'}(T_1+1)}{r_{i_1'}}\rceil\rceil := T_2$.

If $i_1' \in \underline{A}(0)$, similar to case I we get $|\underline{A}(T_2+1)| = |\underline{A}(0)| - 1$.

If $i_1' \notin \underline{A}(0)$, similar to (A.5) we have

$$|\mathcal{L}_l(T_2+1)| \le |\mathcal{L}_l(T_1+1)| - 1. \tag{A.6}$$

Repeat the above process until the agents in $C_i(x(0))$ split into different maximal-confidence clusters, or $|\underline{A}(T_p+1)| = |\underline{A}(0)| - 1$ for some positive integer p. By (A.5)-(A.6) we get that

$$p \le |\mathcal{L}_l(0)| - |\underline{A}(0)| + 1 \le |C_i(x(0))| - |\underline{A}(0)|.$$

From this and the definition of $T_1, T_2, \ldots$ we have

$$T_p + 1 \\ \le (|C_i(x(0))| - |\underline{A}(0)|)\left(1 + \lceil\log_2\lceil r_{\max}^i / r_{\min}^i\rceil\rceil\right).$$

Let t_1 be the minimal time such that E1 or E2 happens. By the discussion in Cases I and II we have

$$t_1 \le (|C_i(x(0))| - |\underline{A}(0)|)\left(1 + \lceil\log_2\lceil r_{\max}^i / r_{\min}^i\rceil\rceil\right) \tag{A.7}$$

If E1 happens at time t_1, our result i) holds; otherwise, we need to carry out next step.

Step 2: For $s \ge t_1$ we control the agent l moves toward the right until E1 or one of the following two events happens:
(E3) $x_l(s) \ge x_j(0) + r_{\min}^i/2$;
(E4) $\max_{m\in C_i(x(0))} x_m(s) \le x_k(0) - r_{\min}^i/4$;

For $s \ge t_1$, let i_s' be the agent in $C_i(x(0))$ which has the biggest opinion within the confidence bound of agent l at time s, i.e.,

$$i_s' = \arg\max_{m\in C_i(x(0))} \{x_m(s) : |x_l(s) - x_m(s)| \le r_l\}.$$

Choose (i_s', l) as the agent pair for opinion updation, until at least one of the events E1, E3, and E4 happens. Let t_2 be the minimal time that E1, E3, or E4 happens. For $s \in [t_1, t_2)$, since E1 and E4 do not happen at time s,

$$x_l(s+1) = \frac{x_l(s) + x_{i_s'}(s)}{2} > x_l(s).$$

By the similar method as Step 1, each agent in $C_i(x(0)) \backslash (\underline{A}(t_1) \cup \{l\})$ can be chosen at most $1 + \lceil\log_2\lceil r_{\max}^i / r_{\min}^i\rceil\rceil$ times for opinion updation during $[t_1, t_2)$. Then,

$$t_2 - t_1 \\ \le (|C_i(x(0))| - |\underline{A}(t_1)| - 1)\left(1 + \lceil\log_2\lceil r_{\max}^i / r_{\min}^i\rceil\rceil\right) \\ = (|C_i(x(0))| - |\underline{A}(0)|)\left(1 + \lceil\log_2\lceil r_{\max}^i / r_{\min}^i\rceil\rceil\right). \tag{A.8}$$

If E4 happens, by Lemma 7 we have

$$\max_{M,m\in C_i(x(0))} [x_M(t_2) - x_m(t_2)] \\ \le \max_{M\in C_i(x(0))} x_M(t_2) - x_j(0) \le x_k(0) - x_j(0) - r_{\min}^i/4,$$

which indicates our result ii) holds; if E1 happens, our result i) holds at time t_2; otherwise, we need to carry out next Step.

... ...

Step $2m+1$: For $s \ge t_{2m}$, we use the similar control method as Step 1. Let t_{2m+1} be the minimal time such that E1 happens or $|\underline{A}(t_{2m+1})| = |\underline{A}(t_{2m-1})| - 1$. Similar to (A.7) we have

$$t_{2m+1} - t_{2m} \\ \le (|C_i(x(0))| - |\underline{A}(t_{2m-1})|)\left(1 + \lceil\log_2\lceil r_{\max}^i / r_{\min}^i\rceil\rceil\right) \\ = (|C_i(x(0))| - |\underline{A}(0)| + m)\left(1 + \lceil\log_2\lceil r_{\max}^i / r_{\min}^i\rceil\rceil\right). \tag{A.9}$$

Step $2m+2$: For $s \ge t_{2m+1}$, we use the similar control method as Step 2. Let t_{2m+2} be the minimal time such that E1, E3, or E4 happens. Similar to (A.8) we have

$$t_{2m+2} - t_{2m+1} \\ \le (|C_i(x(0))| - |\underline{A}(t_{2m+1})| - 1)\left(1 + \lceil\log_2\lceil r_{\max}^i / r_{\min}^i\rceil\rceil\right) \\ = (|C_i(x(0))| - |\underline{A}(0)| + m)\left(1 + \lceil\log_2\lceil r_{\max}^i / r_{\min}^i\rceil\rceil\right). \tag{A.10}$$

The above process will end at Step $2|\underline{A}(0)| - 1$ because $\underline{A}(t_{2|\underline{A}(0)|-1}) = \emptyset$. By Lemma 7 and the definition of

$\underline{A}(s)$ we have

$$\begin{aligned}
&\max_{M,m\in C_i(x(0))}[x_M(t_{2|\underline{A}(0)|-1}) - x_m(t_{2|\underline{A}(0)|-1})] \\
&\leq x_k(0) - \min_{m\in C_i(x(0))} x_m(t_{2|\underline{A}(0)|-1}) \qquad \text{(A.11)} \\
&\leq x_k(0) - x_j(0) - r^i_{\min}/4,
\end{aligned}$$

which indicates our result ii) holds when $t^* = t_{2|\underline{A}(0)|-1}$. Set $t_0 := 0$. By (A.10) and (A.11) we have

$$\begin{aligned}
&t_{2|\underline{A}(0)|-1} \\
&= \sum_{m=0}^{|\underline{A}(0)|-2} (t_{2m+2} - t_{2m}) + t_{2|\underline{A}(0)|-1} - t_{2|\underline{A}(0)|-2} \\
&\leq \Bigg(\sum_{m=0}^{|\underline{A}(0)|-2} 2\left(|C_i(x(0))| - |\underline{A}(0)| + m\right) \\
&\quad + |C_i(x(0))| - 1\Bigg) \left(1 + \lceil \log_2 \lceil r^i_{\max}/r^i_{\min}\rceil\rceil\right) \\
&= \big((2|\underline{A}(0)| - 1)|C_i(x(0))| + (-|\underline{A}(0)| - 1)|\underline{A}(0)| + 1\big) \\
&\quad \cdot \left(1 + \lceil \log_2 \lceil r^i_{\max}/r^i_{\min}\rceil\rceil\right) \\
&\leq (|C_i(x(0))| - 1)^2 \left(1 + \lceil \log_2 \lceil r^i_{\max}/r^i_{\min}\rceil\rceil\right),
\end{aligned}$$

where the last inequality uses the fact that $|\underline{A}(0)| \leq |C_i(x(0))| - 1$.

For the case when $x_l(0) < \frac{x_k(0)+x_j(0)}{2}$, we can set

$$\bar{A}(s) := \{m \in C_i(x(0)) : x_m(s) > x_k(0) - r^i_{\min}/4\},$$

and use the similar method as the case $x_l(0) \geq \frac{x_k(0)+x_j(0)}{2}$ to control $\bar{A}(s)$ becomes empty. $\square$