Dynamic Models of Appraisal Networks Explaining Collective Learning

Wenjun Mei, Noah E. Friedkin, Kyle Lewis, Francesco Bullo

***Abstract*— This paper proposes models of learning process in groups of individuals who collectively execute a sequence of tasks and whose actions are determined by individual skill levels and networks of interpersonal appraisals and influence. The closely-related proposed models have increasing complexity, starting with a centralized manager-based assignment and learning model, and finishing with a social model of interpersonal appraisal, assignments, learning, and influences. We show how rational optimal behavior arises along the task sequence for each model. Our models are grounded in replicator dynamics from evolutionary games, influence networks from mathematical sociology, and transactive memory systems from organization science.**

***Index Terms*— collective learning, transactive memory system, appraisal network, influence network, evolutionary games, replicator dynamics, multi-agent systems**

I. Introduction

a) Motivation and problem description: Researchers in sociology, psychology, and organization science have long studied the inner functioning and performance of teams with multiple individuals engaged in tasks. Extensive qualitative studies, conceptual models and empirical studies reveal some statistical features and various phenomena of teams [1], [2], [3], [4], [5], but only a few quantitative and mathematical models are available [6], [7], [8]. In this paper we build mathematical models for the dynamics of team structure and performance. Our work is based on the core idea that the exhibited phenomena and features of teams must result from some essential elements, that is, individual (member) attributes and the team's inner structure. We aim to mathematically characterize these essential elements, investigate how they relate to team performance, and, most importantly, how they evolve with time.

We consider a team of individuals with unknown skill levels who complete a sequence of tasks. The team's basic inner structure is characterized by the appraisal network, which determines how the task is assigned, and is updated via performance feedback and the co-evolution with the influence network. We aim to build multi-agent dynamical models in which (i) the team as an entirety eventually achieves the optimal task assignment; (ii) each individual's true relative skill level is asymptotically learned by the team members, which is referred to as collective learning.

b) Literature review: Our work is deeply connected with a conceptual models of teams, the *transactive memory system* (TMS), which generates a shared division of cognitive labor with respect to the encoding, storage, retrieval and communication of information [9]. The TMS of a team is linked with the individual and team performance [10], [11]. Lewis [1] describes the behavioral indicators that a TMS is operating in a group: the degree to which members specialize in complementary yet distinct aspects of the groups tasks (specialization), the extent that members rely on the expertise of other members (credibility), and evidence of coordinated interdependent activity (coordination).

In our models, collective learning arises as the result of the co-evolution of the interpersonal appraisals and influence networks. Related previous work includes social comparison theory [13], averaging-based social learning [14], opinion dynamics on influence networks [15], [16], [17], reflected appraisal mechanisms [18], [19], [20], dynamic balance theory [21], [22], [23], and the combined evolution of interpersonal appraisals and influence networks [24].

In the modeling and analysis of the evolution of appraisal and influence networks, we also build an insightful connection between our model and the well-known replicator dynamics studied in evolutionary game theory; see the textbook [25], some control applications [26], [27], and the recent contribution [28].

c) Contribution: Firstly, based on a few natural assumptions, we propose three novel models on dynamics of teams: manager dynamics, assign/appraise dynamics, and assign/appraise/influence dynamics, with increasing complexity. Our work integrates three well-established types of dynamics: replicator dynamics, dynamics of appraisal networks, and opinion dynamics on influence networks. To the best of our knowledge, this is the first time that such an integration has been proposed. Moreover, our models provide an innovative perspective on the connection between team performance and appraisal network. In our models, performances of a group of individuals, with fixed skill levels, are determined by how the task is assigned. In the baseline manager dynamics, task assignment is determined by an outside authority, and adjusted via the replicator dynamics with individuals' performance feedback. The assign/appraise dynamics elaborates the baseline model by assuming that, individuals' interpersonal appraisals, instead of an outsider authority, determine the task assignment. In the assign/appraise/influence dynamics model, we further

This material is based upon work supported by, or in part by, the U. S. Army Research Laboratory and the U. S. Army Research Office under grant numbers W911NF-15-1-0577 and W911NF-16-1-0005. The content of the information does not necessarily reflect the position or the policy of the Government, and no official endorsement should be inferred.

Wenjun Mei and Francesco Bullo are with the Center for Control, Dynamical Systems, and Computation, University of California, Santa Barbara, Santa Barbara, CA 93106, USA, meiwenjunbd@gmail.com, bullo@engineering.ucsb.edu Noah E. Friedkin is with the Department of Sociology, University of California, Santa Barbara, Santa Barbara, CA 93106, USA, friedkin@soc.ucsb.edu. Kyle Lewis is with the Technology Management Program, University of California, Santa Barbara, Santa Barbara, CA 93106, USA, klewis@tmp.ucsb.edu

elaborate the assign/appraise dynamics by considering the co-evolution of appraisal and influence networks.

Secondly, theoretical analysis is presented on the dynamical properties of the models we propose. One of our key results is that, for assign/appraise dynamics and assign/appraise/influence dynamics, the left dominant eigenvector (a well-established centrality notion) of the appraisal matrix obeys the replicator dynamics (with a multiplicative bias), as in the manager dynamics. We also establish that, for each of the models we propose, task assignment converges to its optimal value. Moreover, for assign/appraise/influence dynamics, every individual asymptotically learns the true relative skill levels of all the team members.

Thirdly, we remark that, theoretical results on assign/appraise/influence dynamics reveal a further connection with the TMS theory. According to the work by Palazzolo [29] and Lee et al. [30], internal cognitive consistence is positively correlated with mature TMS. The assign/appraise/influence dynamics also describe an emergence process by which team members' perception of "who knows what" become more similar over time.

d) Organization: The rest of this paper is organized as follows: Section II introduces some preliminaries on evolutionary games and replicator dynamics; Section III proposes our problem set-up and centralized manager model; Section IV introduces the assign/appraise dynamics; Section V introduces the assign/appraise/influence model; Section VI provides some further discussions and conclusion. We postpone all proofs to a forthcoming journal submission.

II. Preliminaries

Evolutionary games apply game theory to evolving populations adopting different strategies. Consider a game with finite pure strategies $\boldsymbol{e}_1, \boldsymbol{e}_2, \dots, \boldsymbol{e}_n$ and mixed strategies $\boldsymbol{w} \in \operatorname{int}(\Delta_n)$. The expected payoff for mixed strategy $\boldsymbol{v}$ against mixed strategy $\boldsymbol{w}$ is defined as the *payoff function* $\pi(\boldsymbol{v}, \boldsymbol{w}) = \sum_{i=1}^n v_i \pi_i(\boldsymbol{w})$, with $\pi_i(\boldsymbol{w}) = \pi(\boldsymbol{e}_i, \boldsymbol{w})$ for simplicity. A strategy $\hat{\boldsymbol{w}}$ is an *evolutionary stable strategy* (ESS) if any mutant strategy $\boldsymbol{v} \neq \hat{\boldsymbol{w}}$, adopted by an ϵ-fraction of the population, brings less expected payoff than the majority strategy $\hat{\boldsymbol{w}}$, as long as ϵ is sufficiently small. A necessary and sufficient condition for a local ESS is stated as follows: there exists a neighborhood $U(\hat{\boldsymbol{w}})$ such that, for any $\boldsymbol{v} \in U(\hat{\boldsymbol{w}}) \setminus \{\hat{\boldsymbol{w}}\}$, $\pi(\hat{\boldsymbol{w}}, \boldsymbol{v}) > \pi(\boldsymbol{v}, \boldsymbol{v})$.

Replicator dynamics, given by equation (1), models the evolution of sub-population distribution $\boldsymbol{w}(t) \in \Delta_n$. Each sub-population i, with fraction $w_i(t)$ at time t, is using strategy $\boldsymbol{e}_i$ and has the growth rate proportional to its *fitness*, defined as the expected payoff $\pi_i(\boldsymbol{w}(t))$.

$$\dot{w}_i = w_i\Big(\pi_i(\boldsymbol{w}) - \sum_{k=1}^n w_k \pi_k(\boldsymbol{w})\Big). \tag{1}$$

The time index t is omitted for simplicity. There is a simple connection between the ESS and the replicator dynamics [25]: if the payoff function $\pi(\boldsymbol{v}, \boldsymbol{w})$ is linear to $\boldsymbol{w}$, then an ESS is a globally asymptotically stable equilibrium for system (1); if $\pi(\boldsymbol{v}, \boldsymbol{w})$ is nonlinear to $\boldsymbol{w}$, then the ESS is locally asymptotically stable.

III. Problem Set-up and Manager Dynamics

In this section we introduce some basic formulations and a baseline centralized system on the evolution of a team. Frequently used notations are listed in Table I.

A. Model assumptions and notations

The assumption on individuals and tasks are given below.

Assumption 1 (Team, task type and assignment): Consider a team of n individuals characterized by a fixed but unknown vector $\boldsymbol{x} = (x_1, \dots, x_n)^\top$ satisfying $\boldsymbol{x} \succ \mathbb{0}_n$ and $\boldsymbol{x}^\top \mathbb{1}_n = 1$, where each x_i denotes the *skill level* of individual i. The tasks being completed by the team are assumed to have the following properties:

(i) The total workload of each task is characterized by a positive scalar and is fixed as 1 in this paper;

(ii) The task can be arbitrarily decomposed into n sub-tasks according to the *task assignment* $\boldsymbol{w} = (w_1, \dots, w_n)^\top$, where each w_i is the sub-task workload assigned to individual i. The task assignment satisfies $\boldsymbol{w} \succ \mathbb{0}_n$ and $\boldsymbol{w}^\top \mathbb{1}_n = 1$. The sub-tasks are executed simultaneously.

The scalar setting of skill levels and task assignments can be simply interpreted as the assumption that, the type of tasks considered in this paper only requires some one-dimension skill. Alternatively, in a more general way, the skill levels can be considered as individuals' overall abilities of contributing to the completion of tasks, and the task assignments are the individuals' relative responsibilities to the team, which forms naturally in the process of completing the tasks.

With fixed skill levels $\boldsymbol{x}$, the measure of each individual i's performance is assumed to be only a function of $\boldsymbol{w}$, defined by the following assumption.

Assumption 2 (Individual performance): Given fixed $\boldsymbol{x}$, each individual i's performance, with the assignment $\boldsymbol{w}$, is measured by $p_i(\boldsymbol{w}) = f(x_i/w_i)$, where $f: \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is continuously differentiable and monotonically increasing.

Despite the specific form $f(x_i/w_i)$ as in Assumption 2, the measure of individual performance can be quite general by adopting difference measures of x_i and w_i.

It is reasonable to claim that, in a well-functioning team, individuals' relative responsibilities, characterized by the task assignment in this paper, should be proportional to their actual abilities. Define the measure of the mismatch between task assignment and individual's true skill levels as $\mathcal{H}_1(\boldsymbol{w}) = \sum_{i=1}^n |w_i/x_i - 1|$. For fixed $\boldsymbol{x}$, the *optimal assignment* $\boldsymbol{w}^* = \boldsymbol{x}$ minimizes $\mathcal{H}_1(\boldsymbol{w})$.

B. Centralized manager dynamics

In this subsection we introduce a continuous-time centralized model on the evolution of task assignment, referred to as the *manager dynamics*. The diagram illustration is given by Figure 1(a). Suppose that, at time t, a team of n individuals is completing a task based on the assignment $\boldsymbol{w}(t)$. The manger

TABLE I

NOTATIONS FREQUENTLY USED IN THIS PAPER

$\succ$ ($\prec$ resp.)	entry-wise strictly greater than (less than resp.).
$\succeq$ ($\preceq$ resp.)	entry-wise no less than (no greater than resp.).
$\mathbb{1}_n$ ($\mathbb{0}_n$ resp.)	n-dimension column vector with all entries equal to 1 (0 resp.)
$\boldsymbol{x}$	vector of individual skill levels, with $\boldsymbol{x} = (x_1, x_2, \dots, x_n)^\top \succ \mathbb{0}_n$ and $\boldsymbol{x}^\top \mathbb{1}_n = 1$.
$\boldsymbol{w}$	task assignment. $\boldsymbol{w} \succ \mathbb{0}_n$ and $\boldsymbol{w}^\top \mathbb{1}_n = 1$
f	a concave, continuously differentiable and increasing function f: $[0, +\infty) \to [0, +\infty)$
$\boldsymbol{p}(\boldsymbol{w})$	vector of individual performances. $\boldsymbol{p}(\boldsymbol{w}) = \big(p_1(\boldsymbol{w}), \dots, p_n(\boldsymbol{w})\big)^\top$, where $p_i(\boldsymbol{w}) = f(w_i/x_i)$ is the performance of individual i.
A	appraisal matrix. $A = (a_{ij})_{n\times n}$, where a_{ij} is individual i's appraisal of j's skill level.
W	influence matrix. $W = (w_{ij})_{n\times n}$, where w_{ij} is the weight individual i assigns to j's opinion.
Δ_n	n-dimension simplex $\{\boldsymbol{y} \in \mathbb{R}^n_{\geq 0} \,\vert\, \boldsymbol{y}^\top \mathbb{1}_n = 1\}$
$\operatorname{int}(\Delta_n)$	the interior of Δ_n.
$\boldsymbol{v}_{\text{left}}(A)$	the left dominant eigenvector of the non-negative and irreducible matrix A, i.e., the normalized entry-wise positive left eigenvector associated with the eigenvalue equal to A's spectral radius.

observes the individuals' real-time performance $\boldsymbol{p}\big(\boldsymbol{w}(t)\big)$ and adjust the task assignment $\boldsymbol{w}(t)$ according to the dynamics:

$$\dot{w}_i = w_i\Big(p_i(\boldsymbol{w}) - \sum_{k=1}^n w_k p_k(\boldsymbol{w})\Big), \tag{2}$$

for any $i \in \{1, \dots, n\}$. The following theorem states the asymptotic behavior of the manager dynamics.

Theorem 1 (Behavior of manager dynamics): Consider the manager dynamics (2) for the task assignment as in Assumption 1 with performance as in Assumption 2. Then

(i) the set $\operatorname{int}(\Delta_n)$ is invariant;

(ii) the optimal assignment $\boldsymbol{w}^* = \boldsymbol{x}$ is the ESS for the evolutionary game defined by the payoff function $\pi(\boldsymbol{v}, \boldsymbol{w}) = \sum_{i=1}^n v_i f(x_i/w_i)$, and is thus a locally asymptotically stable equilibrium for equation (2);

(iii) for any $\boldsymbol{w}(0) \in \operatorname{int}(\Delta_n)$, the manager's assignment $\boldsymbol{w}(t)$ converges to $\boldsymbol{w}^* = \boldsymbol{x}$, as $t \to \infty$.

Equation (2) takes the same form as the classic replicator dynamics [25], with the nonlinear fitness function $\pi_i(\boldsymbol{w}) = f(x_i/w_i)$. Distinct from the classic result that, the ESS with nonlinear payoff function can only lead to local asymptotic stability for the replicator dynamics, our model is a special case in which the ESS associated with a nonlinear payoff function is also a globally asymptotically stable equilibrium of the replicator dynamics.

IV. THE ASSIGN/APPRAISE DYNAMICS OF APPRAISAL NETWORKS

Despite the desired property on the convergence of task assignment to optimality, the manger dynamics does not capture one of the most essential aspects of team dynamics: the evolution of the team's inner structures. In this section, we introduce a multi-agent system, elaborated from the baseline manager dynamics, in which the team members' interpersonal appraisals, rather than any manager, determine the task assignment, and the appraisal network is updated via performance feedback signal observed by each team member.

A. Model description and problem statement

Appraisal network: Denote by a_{ij} the individual i's evaluation of j's skill levels and refer to $A = (a_{ij})_{n\times n}$ as the *appraisal matrix*. Since the evaluations are in the relative sense, we assume $A \succeq \mathbb{0}_{n\times n}$ and $A\mathbb{1}_n = \mathbb{1}_n$. The directed and weighted graph $G(A)$, referred to as the *appraisal network*, reflects the team's collective knowledge on the distribution of its members' abilities.

Assign/appraise dynamics: We propose a multi-agent model on the evolution of the appraisal network. The model is referred to as the *assign/appraise dynamics* and illustrated by the diagram in Figure 1(b). We model three phases: task assignment, feedback signal, and update of appraisal network, specified by the following three assumptions respectively.

Assumption 3 (Assignment rule): At any time $t \geq 0$, a task is divided and assigned according to the left dominant eigenvector of the appraisal matrix, i.e., $\boldsymbol{w}(t) = \boldsymbol{v}_{\text{left}}\big(A(t)\big)$.

For now we assume $A(t)$ is row-stochastic and irreducible for all $t \geq 0$, so that $\boldsymbol{v}_{\text{left}}\big(A(t)\big)$ is always well-defined. We will present a theorem on the well-definedness of the assignment later in this section.

Assumption 4 (Feedback signal): After executing the task with assignment $\boldsymbol{w}$, each individual i observes, without any noise, the difference between her own performance $p_i(\boldsymbol{w})$ and the team's weighted average performance, given by $p_{\text{ave}}(\boldsymbol{w}) = \sum_{k=1}^n w_k p_k(\boldsymbol{w})$.

Assumption 5 (Update of interpersonal appraisals): With performance feedback signal defined as in Assumption 4, each individual i increases her self appraisal and decreases the appraisals of all the other individuals, if $p_i(\boldsymbol{w}) > p_{\text{ave}}(\boldsymbol{w})$, and vice versa. In addition, the appraisal matrix $A(t)$ remains row-stochastic.

The following dynamical system for the appraisal matrix, referred to as the *appraise dynamics*, is arguably the simplest model satisfying Assumptions 4 and 5:

$$\begin{cases} \dot{a}_{ii} = a_{ii}(1 - a_{ii})\Big(p_i(\boldsymbol{w}) - p_{\text{ave}}(\boldsymbol{w})\Big), \\ \dot{a}_{ij} = -a_{ii}a_{ij}\Big(p_i(\boldsymbol{w}) - p_{\text{ave}}(\boldsymbol{w})\Big). \end{cases} \tag{3}$$

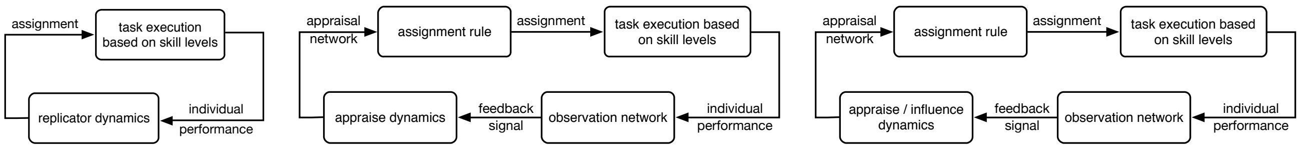
The matrix form of the appraise dynamics, together with the assignment rule as in Assumption 3, is given by

$$\begin{cases} \dot{A} = \operatorname{diag}\big(\boldsymbol{p}(\boldsymbol{w}) - p_{\text{ave}}(\boldsymbol{w})\mathbb{1}_n\big)A_{\text{d}}(I_n - A), \\ \boldsymbol{w} = \boldsymbol{v}_{\text{left}}(A), \end{cases} \tag{4}$$

and collectively referred to as the assign/appraise dynamics. Here $A_{\text{d}} = \operatorname{diag}(a_{11}, \dots, a_{nn})$.

Problem statement: In the next subsection, we investigate the asymptotic behavior of dynamics (4), including:

(i) convergence to the optimal assignment, which means that the team as an entirety eventually learns its members' relative skill levels, i.e., $\lim_{t\to+\infty} \boldsymbol{w}(t) = \boldsymbol{x}$;



(a) manager dynamics (b) assign/appraise dynamics (c) assign/appraise/influence dynamics

Fig. 1. Diagram illustrations of manager dynamics, assign/appraise dynamics, and assign/appraise/influence dynamics.

(ii) *appraisal consensus*, which means that the individuals asymptotically reach consensus on the appraisals of all the team members, i.e., $a_{ij}(t) - a_{kj}(t) \to 0$ as $t \to +\infty$, for any i, j, k.

Collective learning is the combination of convergence to the optimal assignment and appraisal consensus.

B. Dynamical behavior of the assign/appraise dynamics

We start by establishing that the appraisal matrix $A(t)$, as the solution to equation (4), is extensible to all $t \in [0, +\infty)$ and the assignment $\boldsymbol{w}(t)$ is well-defined, in that $A(t)$ remains row-stochastic and irreducible. Moreover, some finite-time properties are investigated.

Theorem 2 (Well-definedness and finite-time properties): Consider the assign/appraise dynamics (4), based on Assumptions 3-5, describing a task assignment as in Assumption 1, with performance as in Assumption 2. For any initial appraisal matrix that is row-stochastic, irreducible and has strictly positive diagonal,

(i) The appraisal matrix $A(t)$, as the solution to (4), is extensible to all $t \in [0, +\infty)$. Moreover, $A(t)$ remains row-stochastic, irreducible and has strictly positive diagonal for all $t \geq 0$;

(ii) there exists a row-stochastic irreducible matrix $C \in \mathbb{R}^{n \times n}$ with zero diagonal such that

$$A(t) = \operatorname{diag}\big(\boldsymbol{a}(t)\big) + \big(I_n - \operatorname{diag}\big(\boldsymbol{a}(t)\big)\big)\, C, \tag{5}$$

for all $t \geq 0$, where $\boldsymbol{a}(t) = \big(a_1(t), \dots, a_n(t)\big)^\top$ and $a_i(t) = a_{ii}(t)$, for $i \in \{1, \dots, n\}$;

(iii) Define the *reduced assign/appraise dynamics* as

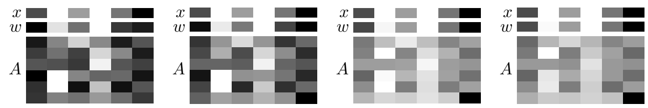
$$\begin{cases} \dot{a}_i = a_i(1 - a_i)\big(p_i(\boldsymbol{w}) - p_{\text{ave}}(\boldsymbol{w})\big), \\ w_i = \dfrac{c_i}{(1 - a_i)} \Big/ \displaystyle\sum_{k=1}^{n} \dfrac{c_k}{(1 - a_k)}, \end{cases} \tag{6}$$

where $\boldsymbol{c} = (c_1, \dots, c_n)^\top = \boldsymbol{v}_{\text{left}}(C)$. This dynamics is equivalent to system (4) in the following sense: The matrix $A(t)$'s each diagonal entry $a_{ii}(t)$ satisfies the dynamics (6) for $a_i(t)$, and, for any $t \geq 0$, $a_{ii}(t) = a_i(t)$ for any i, and $a_{ij}(t) = a_{ij}(0)\big(1 - a_i(t)\big)/\big(1 - a_i(0)\big)$ for any $i \neq j$;

(iv) The set $\Omega = \big\{\boldsymbol{a} \in [0,1]^n \big| 0 \leq a_i \leq 1 - \zeta_i\big(\boldsymbol{a}(0)\big)\big\}$, where $\zeta_i\big(\boldsymbol{a}(0)\big) = \frac{c_i}{x_i} \min_k \frac{x_k}{c_k}\big(1 - a_k(0)\big)$, is a compact positively invariant set for the reduced assign/appraise dynamics (6);

(v) the assignment $\boldsymbol{w}(t)$ satisfies the *generalized replicator dynamics* with time-varying fitness function $a_i(t)\Big(p_i\big(\boldsymbol{w}(t)\big) - p_{\text{ave}}\big(\boldsymbol{w}(t)\big)\Big)$ for each i:

$$\begin{aligned} \dot{w}_i = w_i\Big(& a_i\big(p_i(\boldsymbol{w}) - p_{\text{ave}}(\boldsymbol{w})\big) \\ & - \sum_{k=1}^{n} w_k a_k\big(p_k(\boldsymbol{w}) - p_{\text{ave}}(\boldsymbol{w})\big)\Big). \end{aligned} \tag{7}$$



(a) $t = 0$ (b) $t = 2$ (c) $t = 10$ (d) $t = 30$

Fig. 2. Visualization of the evolution of $A(t)$ and $\boldsymbol{w}(t)$ obeying the assign/appraise dynamics with $n = 6$. In these visualized matrices and vectors, the darker the entry, the higher value it has.

With the extensibility of $A(t)$ and the finite-time properties, we now present the main theorem of this section.

Theorem 3 (Asymptotic behavior): Consider the assign/appraise dynamics (4), based on Assumptions 3-5, with the task assignment as in Assumption 1 and the performance as in Assumption 2. For any initial appraisal matrix that is irreducible, row-stochastic, and has positive diagonal,

(i) the solution $A(t)$ converges, i.e., there exists $A^* \in \mathbb{R}^{n \times n}$ such that $\lim_{t \to \infty} A(t) = A^*$;

(ii) the limit appraisal matrix A^* is row-stochastic and irreducible. Moreover, the task assignment satisfies $\lim_{t \to \infty} \boldsymbol{w}(t) = \boldsymbol{v}_{\text{left}}(A^*) = \boldsymbol{x}$.

This theorem indicates that, the teams obeying the assign/appraise dynamics asymptotically achieves the optimal task assignment, but do not necessarily reach appraisal consensus. Figure 2 gives a visualized illustration of the asymptotic behavior of the assign/appraise dynamics.

V. The Assign/appraise/influence Dynamics of Appraisal Networks

In this section we further elaborate the assign/appraise dynamics by assuming that, the appraisal network is updated via not only the performance feedback, but also the co-evolution with influence network.

A. Model description

The new model, referred to as the *assign/appraise/influence dynamics*, is defined by three components: the assignment rule as in Assumption 3, the appraise dynamics based on Assumptions 4 and 5, and the *influence dynamics*, which is the opinion exchanges among individuals on interpersonal appraisals. Diagram illustration of assign/appraise/influence dynamics is presented in Figure 1(c). Denote by w_{ij} the weight individual i assigns to j (including the self weight w_{ii}) in the opinion exchange. The matrix $W = (w_{ij})_{n\times n}$ defines a directed and weighted graph, referred to as the *influence network*. The construction of influence network and the opinion dynamics all the individuals obey are defined as follows.

Assumption 6 (influence dynamic): Assume that, at each time $t \geq 0$, the influence network is identical to the appraisal network, i.e., $W(t) = A(t)$. Moreover, assume that the individuals obeys the classic DeGroot opinion dynamics [15] on the interpersonal appraisals.

The equations for assign/appraise/influence dynamics are written as

$$\begin{cases} \dot{A} = \frac{1}{\tau_{\text{ave}}}(A^2 - A) \\ \qquad + \frac{1}{\tau_{\text{app}}} \operatorname{diag}\big(\boldsymbol{p}(\boldsymbol{w}) - p_{\text{ave}}(\boldsymbol{w})\mathbb{1}_n\big) A_{\text{d}}(I_n - A), \\ \boldsymbol{w} = \boldsymbol{v}_{\text{left}}(A), \end{cases} \quad (8)$$

The time index t is omitted for simplicity. The first term on the right-hand side of the first equation in (8) corresponds to the influence dynamics, while the second term on the right-hand side corresponds to the appraise dynamics. Parameters τ_{ave} and τ_{app} are positive, and relate to the time scales of influence dynamics and appraise dynamics respectively.

B. Dynamical behavior of the assign/appraise/influence dynamics

First of all, the following lemma shows that, for the assign/appraise/influence dynamics, we only need to consider the all-to-all initial appraisal network.

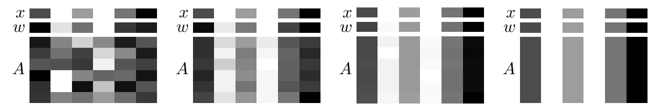
Lemma 4 (Entry-wise strictly positive appraisals): Consider the assign/appraise/influence dynamics (8) based on Assumptions 3-6, with the task assignment and performance as in Assumptions 1 and 2 respectively. For any initial appraisal matrix $A(0)$ that is primitive and row-stochastic, there exists $\Delta t > 0$ such that $A(t) \succ \mathbb{0}_{n\times n}$ for any $t \in (0, \Delta t]$.

Before discussing the asymptotic behavior, we state a technical assumption.

Conjecture 5 (Strict lower bound of appraisals): Consider the assign/appraise/influence dynamics (8) based on Assumptions 3-6, with the task assignment and performance as in Assumptions 1 and 2 respectively. For any $A(0)$ that is entry-wise positive and row-stochastic, there exists $a_{\min} > 0$, depending on $A(0)$, such that $A(t) \succ a_{\min}\mathbb{1}_n\mathbb{1}_n^\top$ for any time $t \geq 0$, as long as $A(\tau)$ and $\boldsymbol{w}(\tau)$ are well-defined for all $\tau \in [0, t]$.

Now we state the main results of this section.

Theorem 6 (Asymptotic behavior): Consider the assign/appraise/influence dynamics (8) based on Assumptions 3-6, with the task assignment and performance as in Assumptions 1 and Assumption 2 respectively. Suppose that Conjecture 5 holds. For any initial appraisal matrix $A(0)$ that is entry-wise strictly positive and row-stochastic,



(a) $t = 0$ (b) $t = 2$ (c) $t = 10$ (d) $t = 30$

Fig. 3. Visualization of the evolution of $A(t)$ and $\boldsymbol{w}(t)$ obeying the assign/appraise/influence dynamics with $n = 6$. In these visualized matrices and vectors, the darker the entry, the higher value it has.

(i) the solution $A(t)$ exists and $\boldsymbol{w}(t) = \boldsymbol{v}_{\text{left}}\big(A(t)\big)$ is well-defined for all $t \in [0, +\infty)$. Moreover, $A(t) \succ \mathbb{0}_{n\times n}$ and $A(t)\mathbb{1}_n = \mathbb{1}_n$ for any $t \geq 0$;

(ii) the assignment $\boldsymbol{w}(t)$ obeys the generalized replicator dynamics (7), and $\xi_0\mathbb{1}_n \preceq \boldsymbol{w}(t) \preceq \big(1 - (n-1)\xi_0\big)\mathbb{1}_n$, where

$$\xi_0 = \left(1 + (n-1)\frac{\max_k x_k}{\min_l x_l}\gamma_0\right)^{-1}, \quad \text{and}$$
$$\gamma_0 = \frac{\max_k x_k/w_k(0)}{\min_l x_l/w_l(0)};$$

(iii) as $t \to +\infty$, $A(t)$ converges to $\mathbb{1}_n\boldsymbol{x}^\top$ and thereby $\boldsymbol{w}(t)$ converges to $\boldsymbol{x}$.

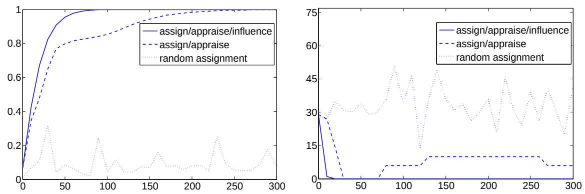
As Theorem 6 indicates, assign/appraise/influence dynamics leads to collective learning. A visualized illustration of the dynamics is given by Figure 3.

VI. Discussion and Conclusion

A. Connections with the TMS theory

TMS structure: As discussed in the introduction, one important aspect of TMS is the members' shared understanding about who possess what expertise. For the case of one-dimension skill, TMS structure is approximately characterized by the appraisal matrix and thus the development of TMS corresponds to the collective learning on individuals' true skill levels. Simulation results in Figure 4 compare the evolution of some features among the teams obeying the assign/appraise/influence model, the assign/appraise model, and the team that randomly assigns the sub-tasks, respectively. Figure 4(a) shows that, for both assign/appraise/influence dynamics and assign/appraise dynamics, function $\mathcal{H}_1(\boldsymbol{w}, \boldsymbol{x})$, as the measure of the mismatch between task assignment and individual skill levels, converge to 0, which exhibits the advantage of a developing TMS.

Transitive triads: As Palazzolo [29] points out, transitive triads are indicative of a well-organized TMS, while non-transitive triads indicate a poorly organized TMS. The underlying is that inconsistency of interpersonal appraisals lowers the efficiency of locating the expertise and allocating the incoming information. In order to reveal the evolution of triad transitivity in our models, we define an unweighted and directed graph, referred to as the *comparative appraisal graph* $\widetilde{G}(A) = (V, E)$, with $V = \{1, \dots, n\}$, as follows: for any $i, j \in V$, $(i, j) \in E$ if $a_{ij} \geq a_{ii}$, i.e., if individual i thinks j has no lower skill level than i herself. We adopt the standard notion of triad transitivity and use the number



(a) $e^{-\mathcal{H}_1(\boldsymbol{w},\boldsymbol{x})}$ (b) Number of non-transitive triads

Fig. 4. Evolution of the measure of mismatch between assignment and individual skill levels, and the number of non-transitive triads in the comparative appraisal graph. The solid curves represent the team obeying the assign/appraise/influence dynamics. The dash curves represent the team obeying the assign/appraise dynamics. The dotted curves represent the team that randomly assign sub-task workloads.

of non-transitive triads as the indicator of inconsistency in a team. Figure 4(b) shows that, the non-transitive triads vanish in the team obeying the assign/appraise/influence dynamics, but persist in the teams obeying the assign/appraise dynamics or just randomly assigning subtasks.

B. Conclusion

This paper proposes a baseline model: the centralized manager dynamics, and two elaborative multi-agent models on team dynamics: the assign/appraise and the assign/appraise/influence dynamics. We reveal insightful connections between our models and the replicator dynamics in evolutionary game theory. For the multi-agents models, the appraisal network is modeled as a team's basic inner structure. The appraisal network generates the team's task assignments, and the mismatch between the assignment and individuals' true skill levels is an indicator of the level of team performance. By theoretical analysis we investigate the asymptotic behavior of the evolution of appraisal network. We also show that the qualitative predictions of our models are consistent with TMS theory in organization science.

References

[1] K. Lewis, "Measuring transactive memory systems in the field: scale development and validation," *Journal of Applied Psychology*, vol. 88, no. 4, p. 587, 2003.

[2] S. Wuchty, B. F. Jones, and B. Uzzi, "The increasing dominance of teams in production of knowledge," *Science*, vol. 316, pp. 1036–1039, 2007.

[3] A. W. Woolley, C. F. Chabris, A. Pentland, N. Hasnmi, and T. W. Malone, "Evidence for a collective intelligence factor in the performance of human groups," *Science*, vol. 330, pp. 686–688, 2010.

[4] W. Mason and D. J. Watts, "Collaborative learning in networks," *Proceedings of the National Academy of Sciences*, vol. 109, no. 3, pp. 764–769, 2011.

[5] S. Hoogendoorn, H. Oosterbeek, and M. van Praag, "The impact of gender diversity on the performance of business teams: Evidence from a field experiment," *Management Science*, vol. 59, pp. 1514–1528, 2013.

[6] J. G. March, "Exploration and exploitation in organizational learning," *Organization Science*, vol. 2, no. 1, pp. 71–87, 1991.

[7] D. Lazer and A. Friedman, "The network structure of exploration and exploitation," *Administrative Science Quarterly*, vol. 52, no. 4, pp. 667–694, 2007.

[8] E. G. Anderson Jr and K. Lewis, "A dynamic model of individual and collective learning amid disruption," *Organization Science*, vol. 25, no. 2, pp. 356–376, 2013.

[9] D. M. Wegner, T. Giuliano, and P. T. Hertel, "Cognitive interdependence in close relationships," in *Compatible and incompatible relationships*, J. Ickes, Ed. Springer, 1985, pp. 253–276.

[10] K. Lewis, "Knowledge and performance in knowledge-worker teams: A longitudinal study of transactive memory systems," *Management Science*, vol. 50, pp. 1519–1533, 2004.

[11] Y. C. Yuan, I. Carboni, and K. Ehrlich, "The impact of awareness and accessibility on expertise retrieval: A multilevel network perspective," *Journal of the American Society for Information Science and Technology*, vol. 61, no. 4, pp. 700–714, 2010.

[12] K. Lewis and B. Herndon, "Transactive memory systems: Current issues and future research directions," *Organization Science*, vol. 22, no. 5, pp. 1254–1265, 2011.

[13] L. Festinger, "A theory of social comparison processes," *Human Relations*, vol. 7, no. 2, pp. 117–140, 1954.

[14] A. Jadbabaie, A. Sandroni, and A. Tahbaz-Salehi, "Non-Bayesian social learning," *Games and Economic Behavior*, vol. 76, no. 1, pp. 210–225, 2012.

[15] M. H. DeGroot, "Reaching a consensus," *Journal of the American Statistical Association*, vol. 69, no. 345, pp. 118–121, 1974.

[16] N. E. Friedkin and C. E. Johnsen, "Attitude change, affect control, and expectation states in the formation of influence networks," *Advances in Group Processes*, vol. 20, pp. 1–29, 2003.

[17] A. MirTabatabaei and F. Bullo, "Opinion dynamics in heterogeneous networks: Convergence conjectures and theorems," *SIAM Journal on Control and Optimization*, vol. 50, no. 5, pp. 2763–2785, 2012.

[18] N. E. Friedkin, "A formal theory of reflected appraisals in the evolution of power," *Administrative Science Quarterly*, vol. 56, no. 4, pp. 501–529, 2011.

[19] P. Jia, A. MirTabatabaei, N. E. Friedkin, and F. Bullo, "Opinion dynamics and the evolution of social power in influence networks," *SIAM Review*, vol. 57, no. 3, pp. 367–397, 2015.

[20] X. Chen, J. Liu, Z. Xu, and T. Başar, "Distributed evaluation and convergence of self-appraisals in social networks," in *IEEE Conf. on Decision and Control*, Osaka, Japan, 2015, pp. 2895–2900.

[21] K. Kułakowski, P. Gawroński, and P. Gronek, "The Heider balance: a continuous approach," *International Journal of Modern Physics C*, vol. 16, no. 05, pp. 707–716, 2005.

[22] S. A. Marvel, J. Kleinberg, R. D. Kleinberg, and S. H. Strogatz, "Continuous-time model of structural balance," *Proceedings of the National Academy of Sciences*, vol. 108, no. 5, pp. 1771–1776, 2011.

[23] V. A. Traag, P. V. Dooren, and P. D. Leenheer, "Dynamical models explaining social balance and evolution of cooperation," *PLOS ONE*, vol. 8, no. 4, p. e60063, 2013.

[24] P. Jia, N. E. Friedkin, and F. Bullo, "The coevolution of appraisal and influence networks leads to structural balance," *IEEE Transactions on Network Science and Engineering*, July 2016, to appear.

[25] W. H. Sandholm, *Population Games and Evolutionary Dynamics*. MIT Press, 2010.

[26] J. R. Marden, H. P. Young, G. Arslan, and J. S. Shamma, "Payoff-based dynamics for multiplayer weakly acyclic games," *SIAM Journal on Control and Optimization*, vol. 48, no. 1, pp. 373–396, 2009.

[27] M. J. Fox, B. Touri, and J. S. Shamma, "Dynamics in atomic signaling games," *Journal of Theoretical Biology*, vol. 376, pp. 82–90, 2015.

[28] D. Madeo and C. Mocenni, "Game interactions and dynamics on networked populations," *IEEE Transactions on Automatic Control*, vol. 60, pp. 1801–1810, 2015.

[29] E. T. Palazzolo, "Organizing for information retrieval in transactive memory systems," *Communication Research*, vol. 32, no. 6, pp. 726–761, 2005.

[30] J. Y. Lee, D. G. Bachrach, and K. Lewis, "Social network ties, transactive memory, and performance in groups," *Organization Science*, vol. 25, no. 3, pp. 951–967, 2014.